Load Balancing Game in Loss Communication Networks

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ABSTRACT

We consider a communication network consisting of m homogeneous source nodes (servers, host computers or processors) and one destination node, connected with pairwise communication lossy links. Self-interested users have flows of packets to pass through communication links and transmit to the destination. The traffic flow of each user can be transmitted via two approaches: a direct path (DP) in which the packet goes directly from the source node arrived at to the destination, and a two-hop indirect path (IP) in which the packet is first relayed to another source node and then takes the direct link from that node to the destination. The packet loss may occur due to network congestion or errors in data transmission over links. Each user makes an action of which route to be selected for transmitting his/her packet, and incurs a loss probability whose value depends on the strategy profile of all users.

When a centralized planner exists, we propose an efficient algorithm for computing the system optimum where the total traffic rate of packets that are successfully transmitted is maximized. In a decentralized environment where distributed decisions are made by the autonomous users continuously, the situation often converges to an equilibrium in which no user can decrease his/her loss probability by unilateral deviation. We thereby provide a full characterization of Nash equilibrium, and study the efficiency loss due to these selfish behaviors, theoretically and empirically.

ACM Reference Format:
Chenhao Wang and Qiong Liu. 2021. Load Balancing Game in Loss Communication Networks. In *Appears at the 3rd Games, Agents, and Incentives Workshop (GAIW 2021). Held as part of the Workshops at the 20th International Conference on Autonomous Agents and Multiagent Systems, London, UK, May 2021*, IFAAMAS, 10 pages.

1 INTRODUCTION

Since the seminal work of Erlang [10], loss networks have been widely studied as mathematical models of various stochastic systems in which different types of resources are used to serve diverse classes of customers involving simultaneous resource possession and nonbacklogging workloads. Various applications include database systems [16], wireless networks [4, 17], and telecommunication systems [15] (see an overview in [11]). For example, in packet-switching telecommunication networks, packet losses may occur either due to buffer overflows (these are congestion losses of packets) or random non-congestion losses that are due to the transmission channel (e.g., a radio channel).

Many works study the routing games or load balancing games in loss networks [2, 3, 5, 23], where selfish users want to balance their workloads on the nodes and lossy links of this network, so as to minimize the amount of packets losses. For example, in distributed computer systems jobs or transactions continuously arrive and need to be processed by computers, in communication networks flows of packets or calls are to transmit through communication links, and transportation flow networks have incoming threads of vehicles to drive through roads, etc. The well studied problem is, how can a centralized authority/planner assign the workloads to maximize the system efficiency, and how the dynamic in a decentralized environment develops.

In this paper, we consider the load balancing in a communication loss network consisting of m source nodes (servers) $\{s_1, \dots, s_m\}$ and one destination node d. Each source s_i has n_i users seeking for service, and the traffic of packets originating from each user is assumed to follow an independent Poisson point process with identical rate ϕ. The nodes in the network are connected by two types of communication links, relay links that connect two sources, and direct links that connect one source to the destination. The relay links have random non-congestion losses with a fixed probability q, and the direct links have congestion losses that depends on the arrival rate and service rate of each server.

The user cannot split her traffic, and has to determine how to route all of her traffic from the source node arrived at to the destination node. There are two approaches for the traffic transmission: a direct path (DP) in which the packet goes directly from the source arrived at to the destination, and a two-hop indirect path (IP) in which the packet is first relayed (forwarded) to another source node and then takes the direct link from that node to the destination.

While the users in classic routing games usually have additive costs (such as delays or tolls) [18, 19], in the loss networks considered, the costs of users are the loss probability of their packets, which is non-additive and even non-convex. Each user wants to minimize the loss probability, and we say a state is *Nash equilibrium* (NE), if no user can decrease her loss probability by unilateral deviation. The concept of NE is of special importance from a dynamic standpoint [18]: in a practical scenario, a user changes its route dynamically and repeatedly, in response to the varying load conditions, and the stability points in such systems are exactly those in which no user can gain by changing.

The most related works are [3, 23]. Both works consider a specific loss network with two source nodes, and each source have equal number of users. Moreover, they only focus on a quite restricted case (i.e., symmetric case), in which the strategies of all users must be the same. They study the global optimization on system efficiency

Appears at the 3rd Games, Agents, and Incentives Workshop (GAIW 2021). Held as part of the Workshops at the 20th International Conference on Autonomous Agents and Multiagent Systems, Aziz, Ceppi, Dickerson, Hosseini, Lev, Mattei, McElfresh, Zick (chairs), May 2021, London, UK. …$ACM ISBN 978-x-xxxx-xxxx-x/YY/MM
…$15.00

and Nash equilibria. We generalize their setting by considering multiple sources and different number of users, and abandoning the restrictions on users' strategies.

Our contributions.

This work contributes to the load balancing game in three aspects: optimizing the system efficiency, characterizing Nash equilibria, and measuring the efficiency loss of Nash equilibria, both theoretically and empirically. As we know, the distributed decisions of selfish agents may result in a solution that is not optimal for the system objectives. The *price of anarchy* (PoA) [14, 22] is a concept that measures how the system efficiency degrades due to strategic behaviors of these selfish agents in a decentralized environment, compared with the system optimum achieved by a centralized authority. Precisely, the PoA in our setting is defined as the worst-case ratio between the total traffic rate in an optimal solution and in a Nash equilibrium.

The formal model and notations are presented in Section 2. Suppose there are m sources and n users (players), each source s_i has n_i users arriving at it. Assume w.l.o.g. $n_1 \geq \cdots \geq n_m$.

We first study a simple setting with two sources (i.e., $m = 2$) in Section 3. We give a threshold (see Theorem 3), say val, so that if n_1 is below val, the system optimum that maximizes the total traffic is achieved when all users choose DP. If n_1 is above val, the system optimum is achieved when all n_2 users in source s_2 choose DP and around val users in s_1 choose DP. Moreover, we provide a characterization of NEs using only the number of users who choose DP in each source, regardless of the identity of users. We also give an upper bound on the PoA, indicating that when n_2 is sufficiently large the PoA approaches to 1.

We investigate the general model with m sources in Section 4. First, we propose an algorithm (Algorithm 1) that finds an optimal solution in $O(mn^2)$ time. The idea heavily relies on Lemma 12, which enables us to focus on finding an index $\tilde{i} \in [m]$, such that no user will choose an IP through any source of $s_1, \ldots, s_{\tilde{i}-1}$, and all users arriving at $s_{\tilde{i}+1}, \ldots, s_m$ choose DP. Second, we give a characterization of NEs (Theorem 15), which depends on not only the number of users selecting DP and IP in each source, but also on which IP is selected. Further, we upper bound the PoA with theoretical analysis.

In Section 5, we implement numerical experiments with a range of transmission loss rate q, a range of service rate μ, and a range of number of users n. All results show very small values of PoA (all below 1.08). Intuitively, when the transmission loss rate decreases or the service rate increases, the total traffic in NEs also increases. These intuitions are also validated by our experiments.

Other related works.

Routing games. As a special class of congestion games, routing games in a network are problems of routing traffic to achieve the best possible network performance, and have been studied within various contexts and within various communities, for example, in the road traffic community [6], the mathematics community [20, 21], the telecommunications [1, 18], and theoretical computer science [7, 8]. The above references have all in common a cost framework which is additive over links, such as delays or tolls, and is flow conserving (the amount entering a node equals the amount leaving it). Routing games with non-additive cost in loss networks are studied in [5, 9, 13].

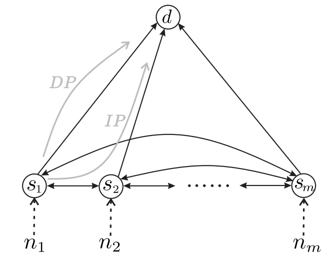

Figure 1: An illustration of the game model.

Braess-like paradox in distributed systems. Braess-like paradox is said to occur in a network system with distributed behaviors if adding an extra link or adding communication capacity to the system leads to a worse system performance. It widely exists in transportation networks and queuing networks. Bean *et al.* [5] show that it can occur in loss networks. Kameda *et al.* [12] consider a model similar to ours in that a job (packet) can be processed directly or indirectly; however, they do not consider the loss probability. They identify a Braess-like paradox in which adding capacity to the channel may degrade the system performance on the *response time.* Kameda and Pourtallier [13] characterize conditions under which such paradoxical behavior occurs, and give examples in which the degradation of performance may increase without bound.

2 MODEL AND PRELIMINARIES

We abstractly model our problem using a graph or network. Consider a network consisting of m *source* nodes $S = \{s_1, \ldots, s_m\}$ and one *destination* node d. For each source node $s_i \in S$, let N_i be the set of users arriving at s_i, and $n_i = |N_i|$ be the number of such users. Denote $[m] = \{1, \ldots, m\}$. There are totally $n = \sum_{i \in [m]} n_i$ users in the system, who are the self-interested players in the game. We say players and users interchangeably throughout this paper. Each user is identified with a flow (or traffic) of packets, which originates from the user and is assumed to form an independent Poisson process with an identical rate ϕ. See Figure 1 for illustration.

Each user controls the route to be followed by all packets of her flow. For a user arriving at $s_i \in S$, there are two types of routes to ship these packets to the destination d: either a direct path (DP) (s_i, d), or an indirect two-hop path (IP) (s_i, s_j, d) for some $s_j \neq s_i$, in which the packet is first sent to another source s_j by the relay link (s_i, s_j), and then passes through the direct link (s_j, d).

Strategies. Each user $k \in N_i$ decides a one-shot *strategy* $\mathbf{p}_k = (p_{k1}, \ldots, p_{km}) \in [0,1]^m$ with $\sum_{j \in [m]} p_{kj} = 1$, where p_{ki} is the probability of routing all packets through DP, and p_{kj} $(j \neq i)$ is the probability of routing all packets through IP (s_i, s_j, d). We focus on *pure strategies* in this paper: a strategy $\mathbf{p}_k$ is *pure* if $\|\mathbf{p}_k\|_\infty = 1$, i.e., user k deterministically selects a route with probability 1. Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ be the strategy profile of all users.

Loss probability and loss rate. The packet loss may occur over both direct links and relay links. The transmission time of a packet in direct link (s, d) for any source $s \in S$ is a random variable σ following a distribution $\mathcal{X}$. Namely, the processing (service) time (without queuing delays) of a packet at any source is σ. The transmission time of different packets are assumed to be independent. We further assume that there is no buffering on these links, so that a packet that enters a direct path during the transmission of another packet is lost. For any relay link (s_i, s_j), a transmission of a packet is lost independently of any other loss with a fixed probability q.

Given strategy profile $\mathbf{p}$, user $k \in N_i$ sends an independent Poisson process of packets with rate $p_{ki}\phi$ to DP (s_i, d), and an independent Poisson process of packets with rate $p_{kj}\phi$ to IP (s_i, s_j, d), for any $s_j \neq s_i$. As a result of the random losses between relay link (s_i, s_j), the flow of packets from user $k \in N_i$ that arrive at node s_j is also a Poisson process with rate $(1-q)p_{kj}\phi$.

Thus, for each source $s_i \in S$, the flow over link (s_i, d) is Poisson with a traffic rate T_i given by

$$T_i(\mathbf{p}) = \sum_{k \in N_i} p_{ki}\phi + \sum_{j \in [m] \setminus \{i\}} \sum_{k \in N_j} p_{ki}(1-q)\phi. \tag{1}$$

When no confusion arises, we simply write $T_i(\mathbf{p})$ as T_i, and other functions defined later may also be simply written without further mention. The probability of no (congestion) loss on the link (s_i, d) equals the probability that there is no arrival during a transmission time σ, which is given by

$$\mathop{\mathbb{E}}_{\sigma \sim \mathcal{X}} e^{-T_i \sigma}.$$

As usual, assume $\mathcal{X}$ is exponential with a rate parameter μ and mean $1/\mu$. Thus, the probability of no loss on (s_i, d) is

$$\mathop{\mathbb{E}}_{\sigma \sim \mathcal{X}} e^{-T_i \sigma} = \int_0^{+\infty} \mu e^{-\mu\sigma} \cdot e^{-T_i \sigma} d\sigma = \frac{\mu}{T_i + \mu},$$

and the loss probability of a packet over link (s_i, d) is $\frac{T_i}{T_i + \mu}$.

Given strategy profile $\mathbf{p}$, for any source $s_i \in S$ and any user $k \in N_i$, the *loss rate* of user k is defined as

$$LR_k(\mathbf{p}) = p_{ki}\phi \frac{T_i}{T_i + \mu} + (1 - p_{ki})q\phi + (1-q)\phi \sum_{j \in [m] \setminus \{i\}} p_{kj} \frac{T_j}{T_j + \mu}, \tag{2}$$

and the loss probability of user k is $\frac{LR_k(\mathbf{p})}{\phi}$.

Total traffic. Regarding the system efficiency, we measure it by the *total traffic rate* arriving at the destination d. Given strategy profile $\mathbf{p}$, the total traffic rate $TR(\mathbf{p})$ of the system has two equivalent expressions. The first expression is based on links:

$$\begin{aligned} TR(\mathbf{p}) &:= \sum_{i \in [m]} T_i \cdot \frac{\mu}{T_i + \mu} \qquad (3) \\ &= \mu \left[m - \sum_{i \in [m]} \frac{\mu}{\sum_{k \in N_i} p_{ki}\phi + \sum_{j \neq i : j \in [m]} \sum_{k \in N_j} p_{ki}(1-q)\phi + \mu} \right]. \end{aligned}$$

where T_i is the traffic rate over link (s_i, d), and $\frac{\mu}{T_i + \mu}$ is the probability of no congestion loss on this link. The second expression is based on users:

$$TR(\mathbf{p}) := \sum_{i \in [m]} \sum_{k \in N_i} (\phi - LR_k), \tag{4}$$

where $\phi - LR_k(\mathbf{p})$ is the traffic rate of user $k \in N_i$ that successfully arriving at d. It is not hard to see that (3) and (4) are equivalent.

Nash equilibria. A Nash equilibrium (NE) is a strategy profile where no player (user) can decrease her loss probability by unilaterally deviating to any other strategy. Formally, we give a definition as follows.

Definition 1. *A strategy profile* $\mathbf{p}$ *is a* Nash equilibrium, *if for any source* $s_i \in S$ *and any player* $k \in N_i$, *we have*

$$LR_k(\mathbf{p}_k, \mathbf{p}_{-k}) \leq LR_k(\mathbf{p}'_k, \mathbf{p}_{-k}),$$

where $\mathbf{p}'_k$ *is any feasible strategy of player* k, *and* $\mathbf{p}_{-k}$ *is the strategy profile of all other players.*

We measure the efficiency of NEs by the price of anarchy (PoA) [14, 22], which is defined as the ratio between social efficiencies in an optimal solution and in the worst NE. Formally, given an instance Γ of this game, define

$$PoA(\Gamma) = \frac{TR(opt)}{\min_{\mathbf{p} \in \mathbb{NE}} TR(\mathbf{p})}.$$

where opt is an optimal solution of Γ, and $\mathbb{NE}$ is the set of all NEs. The PoA of the whole game is defined as the maximum over all instances, that is, $PoA = \max_{\Gamma} PoA(\Gamma)$.

We end this section by an example showing $PoA > 1$.

Example 2. *Consider an instance with* $m = 2$ *sources and* $N_1 = N_2 = 1$, *and* q *is a sufficiently small positive value. Consider the strategy profile* $\mathbf{p}$ *in which both users choose IP. Then the traffic rate over each link is* $(1-q)\phi$, *and by* (3) *the total traffic is* $TR(\mathbf{p}) = 2(1-q)\phi \cdot \frac{\mu}{(1-q)\phi + \mu}$. *By* (2), *the loss probability of each user is* $q + (1-q)\frac{(1-q)\phi}{(1-q)\phi + \mu}$. *If an user deviates her strategy to DP, then her loss probability increases to* $\frac{\phi + (1-q)\phi}{\phi + (1-q)\phi + \mu}$. *So* $\mathbf{p}$ *is a NE. On the other hand, the optimal solution of this instance is that both users choose DP, and by* (3) *the optimal total traffic is* $2\phi \cdot \frac{\mu}{\phi + \mu} > TR(\mathbf{p})$. *Hence, we have* $PoA > 1$.

3 TWO SOURCES

This section is devoted to the special case of two sources (i.e., $m = 2$). Assume w.l.o.g. that $n_1 \geq n_2$. For notational convenience, for each user $i \in N$, we represent her strategy as a single variable $p_i \in \{0, 1\}$, which is the probability of selecting DP. Accordingly, there is a unique IP, and the probability of selecting IP is $1 - p_i$. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be the strategy profile of all users.

We can rewrite the functions defined in Section 2 in a simpler way. Given strategy profile $\mathbf{p}$, define $u_1 = \sum_{i \in N_1} p_i$ to be the number of users in N_1 who choose DP, and $u_2 = \sum_{j \in N_2} p_j$ to be the number of users in N_2 who choose DP. Denote by $p = 1 - q$ the probability that a packet is not lost over a relay link. First, the traffic rate over link (s_1, d) is given by

$$T_1(\mathbf{p}) = \sum_{i \in N_1} p_i\phi + \sum_{j \in N_2} \phi(1 - p_j)p = u_1\phi + (n_2 - u_2)p\phi,$$

and similarly, the traffic rate over link (s_2, d) is

$$T_2(\mathbf{p}) = \sum_{j \in N_2} p_j\phi + \sum_{i \in N_1} \phi(1 - p_i)p = u_2\phi + (n_1 - u_1)p\phi.$$

Next, the loss rate of user $i \in N_1$ is

$$LR_i(\mathbf{p}) = p_i\phi \frac{T_1}{T_1 + \mu} + (1 - p_i)q\phi + p\phi(1 - p_i)\frac{T_2}{T_2 + \mu}, \tag{5}$$

and the loss rate of user $j \in N_2$ is

$$LR_j(\mathbf{p}) = p_j\phi\frac{T_2}{T_2+\mu} + (1-p_j)q\phi + p\phi(1-p_j)\frac{T_1}{T_1+\mu}. \quad (6)$$

Then the total traffic rate is given by

$$TR(\mathbf{p}) = \sum_{i\in N_1}(\phi - LR_i) + \sum_{j\in N_2}(\phi - LR_j) = \frac{T_1\cdot\mu}{T_1+\mu} + \frac{T_2\cdot\mu}{T_2+\mu}. \quad (7)$$

We investigate the system optimum, NEs, and the PoA in the following.

3.1 Computing the optimal solution

As a centralized planner, one wants to maximize the total traffic rate TR arriving at the destination d. Noting that TR can be written as a function of u_1, u_2, the maximization problem is formulated by

$$\max_{u_1,u_2} \frac{T_1\mu}{T_1+\mu} + \frac{T_2\mu}{T_2+\mu} \quad (8)$$

$$s.t.\ 0 \le u_1 \le n_1 \quad (9)$$

$$0 \le u_2 \le n_2 \quad (10)$$

$$u_1, u_2 \in \mathbb{Z}. \quad (11)$$

We simply denote a solution by (u_1, u_2). Given $q < 1$, define

$$val := \frac{n_2 + n_1 p + (1-\sqrt{p})\mu/\phi}{\sqrt{p}+p}.$$

Theorem 3. *If $n_1\sqrt{p} > n_2 + (1-\sqrt{p})\mu/\phi$, then either $(\lfloor val\rfloor, n_2)$ or $(\lceil val\rceil, n_2)$ is an optimal solution. Otherwise, $(u_1^*, u_2^*) = (n_1, n_2)$ is an optimal solution.*

Proof sketch. We first note that in an optimal solution (u_1^*, u_2^*), there is no pair (i, j) of users from (N_1, N_2) who both select IP, as otherwise they can deviate to DP to avoid the loss over delay link and thus improve the total traffic. Since $n_1 \ge n_2$, it must be $u_2^* = n_2$. Then the total traffic rate TR can be written as a function of single variable u_1:

$$TR(u_1) = 2 - \frac{\mu}{T_1+\mu} - \frac{\mu}{T_2+\mu}$$
$$= 2 - \frac{\mu}{u_1\phi+\mu} - \frac{\mu}{n_2\phi+(n_1-u_1)p\phi+\mu}.$$

By computing the derivative, $TR(u_1)$ increases in interval $[0, val]$. Moreover, if $val < n_1$, $TR(u_1)$ decreases in interval $[val, n_1]$. Hence, the maximum is reached when $u_1 = \min\{val, n_1\}$. Taking the integer constraint into account, if $val < n_1$ (i.e., $n_1\sqrt{p} > n_2 + (1-\sqrt{p})\mu/\phi$), the optimal solution (u_1^*, u_2^*) is either $(\lfloor val\rfloor, n_2)$ or $(\lceil val\rceil, n_2)$. If $val \ge n_1$, the optimal solution is (n_1, n_2). □

3.2 Characterization of NEs

We will present a characterization of NEs in this section. Given an arbitrary strategy profile $\mathbf{p}$, define (V_1, V_2, V_3, V_4) as a partition of N, where $V_1 = \{i \in N_1 \mid p_i = 1\}$, $V_2 = \{i \in N_1 \mid p_i = 0\}$, $V_3 = \{j \in N_2 \mid p_j = 1\}$ and $V_4 = \{j \in N_2 \mid p_j = 0\}$. Clearly, all users in V_1 and V_3 choose DP, and all users in V_2 and V_4 choose IP. Supposing $\mathbf{p}$ is a NE, we discuss the deviation of users in V_1, V_2, V_3, V_4 respectively.

For each user $i \in V_1$, by Eq. (5), the loss rate is

$$LR_i(\mathbf{p}) = \frac{\phi T_1(\mathbf{p})}{T_1(\mathbf{p})+\mu} = \phi(1 - \frac{\mu/\phi}{u_1+(n_2-u_2)p+\mu/\phi}).$$

When user $i \in V_1$ deviates to IP, the strategy profile becomes $\mathbf{p}' = (p_i', \mathbf{p}_{-i})$ with $p_i' = 0$, and the loss rate of user i becomes

$$LR_i(\mathbf{p}') = q\phi + p\phi\frac{T_2(\mathbf{p}')}{T_2(\mathbf{p}')+\mu}$$
$$= \phi\left(1 - \frac{p\mu/\phi}{u_2+(n_1-u_1+1)p+\mu/\phi}\right).$$

Since $\mathbf{p}$ is NE, i has no incentive to deviate, and thus $LP_i(\mathbf{p}) \le LR_i(\mathbf{p}')$, which is equivalent to

$$u_1 \le \frac{q\mu/\phi + u_2(1+p^2) + (n_1+1)p - n_2p^2}{2p} := t_1(u_2), \quad (12)$$

where $t_1(u_2)$ is a function with respect to variable u_2.

For each user $i \in V_2$, the loss rate is

$$LR_i(\mathbf{p}) = \phi q + (1-q)\phi\frac{T_2(\mathbf{p})}{T_2(\mathbf{p})+\mu} = \phi\left(1 - \frac{p\mu/\phi}{u_2+(n_1-u_1)p+\mu/\phi}\right).$$

When user $i \in V_2$ deviates to DP, the strategy profile becomes $\mathbf{p}' = (p_i', \mathbf{p}_{-i})$ with $p_i' = 1$, and the loss rate of i becomes

$$LR_i(\mathbf{p}') = \frac{\phi T_1(\mathbf{p}')}{T_1(\mathbf{p}')+\mu} = \phi(1 - \frac{\mu/\phi}{u_1+1+(n_2-u_2)p+\mu/\phi}).$$

It must be $LP_i(\mathbf{p}) \le LR_i(\mathbf{p}')$, which is equivalent to

$$u_1 \ge \frac{q\mu/\phi + u_2(1+p^2) + (n_1-1)p - n_2p^2}{2p} = t_1(u_2) - 1. \quad (13)$$

Symmetrically, for each user $j \in V_3$, since $\mathbf{p}$ is NE, it must be

$$u_2 \le \frac{q\mu/\phi + u_1(1+p^2) + (n_2+1)p - n_1p^2}{2p} := t_2(u_1). \quad (14)$$

For each user $j \in V_4$, it must be

$$u_2 \ge \frac{q\mu/\phi + u_1(1+p^2) + (n_2-1)p - n_1p^2}{2p} = t_2(u_1) - 1. \quad (15)$$

Lemma 4. *Eq. (15) implies Eq. (12), and Eq. (13) implies Eq. (14).*

Proof. By (15), we have $2pu_2 \ge q\mu/\phi + u_1(1+p^2) + (n_2-1)p - n_1p^2$, which is equivalent to that $(1+p^2)u_1 \le 2pu_2 + n_1p^2 - (n_2-1)p - q\mu/\phi$. Because $2pu_2 \le (1+p^2)u_2$ and $n_1p^2 - (n_2-1)p \le (n_1+1)p - n_2p^2$, we have $2pu_1 \le (1+p^2)u_1 \le 2pu_2 + n_1p^2 - (n_2-1)p - q\mu/\phi \le (1+p^2)u_2 + (n_1+1)p - n_2p^2 + q\mu/\phi$, which indicates Eq. (12). By a symmetric analysis, (13) implies (14). □

Now we are ready to give a characterization of NEs.

Theorem 5. *Let $\mathbf{p}$ be an arbitrary strategy profile for the game with two sources. Let u_1 and u_2 be the number of users in N_1 and N_2 who choose DP under $\mathbf{p}$, respectively. We have*

[1] *when (a) $u_1 = n_1, u_2 < n_2$, or (b) $u_1 = 0, u_2 > 0$, $\mathbf{p}$ cannot be a NE;*
[2] *when $u_1 \in [0, n_1), u_2 \in [0, n_2)$, $\mathbf{p}$ is NE if and only if $u_1 \ge t_1(u_2) - 1$ and $u_2 \ge t_2(u_1) - 1$;*
[3] *when $u_1 \in (0, n_1), u_2 = n_2$, $\mathbf{p}$ is NE if and only if $u_1 \in [t_1(u_2) - 1, t_1(u_2)]$;*
[4] *when $u_1 = n_1, u_2 = n_2$, $\mathbf{p}$ is NE if and only if $n_1p \le q\mu/\phi + n_2 + p$.*

Proof. Given $\mathbf{p}$, let (V_1, V_2, V_3, V_4) be a partition of N as defined above. We discuss the four cases.

Case 1. When (a) $u_1 = n_1$ and $u_2 < n_2$, V_4 is nonempty. If $\mathbf{p}$ is a NE, it must satisfy Eq. (15), that is, $2pu_2 \geq q\mu/\phi + n_1(1+p^2) + (n_2-1)p - n_1p^2 = q\mu/\phi + n_1 + (n_2-1)p$. However, because $pu_2 \leq n_1, pu_2 \leq (n_2-1)p$ and $q\mu/\phi > 0$, it cannot hold.

When (b) $u_1 = 0$ and $u_2 > 0$, V_2 is nonempty. If $\mathbf{p}$ is a NE, it must satisfy Eq. (13), that is, $u_1 \geq t_1(u_1) - 1$. It follows that $0 = 2pu_1 \geq q\mu/\phi + u_2(1+p^2) + (n_1-1)p - n_2p^2 \geq q\mu/\phi + 1 + (n_1-1)p - (n_2-1)p^2 \geq q\mu/\phi + 1 > 0$, a contradiction.

Case 2. When $u_1 \in [0, n_1), u_2 \in [0, n_2)$, V_2 and V_4 are nonempty. By Lemma 4, $\mathbf{p}$ is NE if and only if (13) and (15) are satisfied simultaneously.

Case 3. When $u_2 = n_2, u_1 \in (0, n_1)$, V_1, V_2, V_3 are nonempty, and V_4 is empty. $\mathbf{p}$ is NE if and only if (12) (13) and (14) are satisfied simultaneously. By Lemma 4, the sufficiently and necessary condition only needs (12) and (13), that is, $u_1 \in [t_1(u_2)-1, t_1(u_2)]$.

Case 4. When $u_1 = n_1, u_2 = n_2$, V_1, V_3 are nonempty, and V_2, V_4 are empty. $\mathbf{p}$ is NE if and only if (12) and (14) are satisfied, that is, $2pn_1 \leq q\mu/\phi + n_2(1+p^2) + (n_1+1)p - n_2p^2$ and $2pn_2 \leq q\mu/\phi + n_1(1+p^2) + (n_2+1)p - n_1p^2$. It is easy to see that, (12) and (14) are equivalent to $n_1 p \leq q\mu/\phi + n_2 + p$.

Note that every situation of u_1, u_2 is included in the above four cases. So we complete a characterization. □

An intuitive explanation for Case 4 is, when the loss probability q of transmission over link (s_1, s_2) is large enough, no user prefers IP, and the profile that all users select DP is a NE. In addition, when there is no transmission loss over link (s_1, s_2), i.e., $q = 0$, every user prefers a source with fewer users. So the profile that all users select DP is a NE only if the users are balanced on the two sources as evenly as possible, that is, $n_1 \leq n_2 + 1$.

Next, based on Theorem 5, we give some interesting conclusions. Note that the condition $n_1\sqrt{p} \leq n_2 + (1-\sqrt{p})\mu/\phi$ in Theorem 3 implies the condition that $n_1 p \leq q\mu/\phi + n_2 + p$ in Theorem 5 (4), we have the following.

Corollary 6. *If a strategy profile with $u_1 = n_1, u_2 = n_2$ is optimal, then it is also a NE.*

By Theorem 5 (2), a profile with $u_1 = u_2 = 0$ is NE if $0 \geq \frac{q\mu}{\phi} + n_1 - p - n_2p^2$ and $0 \geq \frac{q\mu}{\phi} + n_2 - p - n_1p^2$. Roughly, $n_1 - n_2$ should be at most 1, and the success probability p of transmission should be near 1.

Corollary 7. *A strategy profile with $u_1 = 0, u_2 = 0$ is a NE, if and only if (a) $n_1 = n_2 + 1, p = 1$, or (b) $n_1 = n_2, n_1(1-p^2) \leq p - \frac{q\mu}{\phi}$.*

Note that $u_1 \geq t_1(u_2) - 1$ and $u_2 \geq t_2(u_1) - 1$ cannot hold simultaneously when $q > \frac{2}{n}$, and $u_1 \geq t_1(n_2) - 1$ cannot hold when $n_1 p < q\mu/\phi + n_2 + p$.

Corollary 8. *When $n_1 p < q\mu/\phi + n_2 + p$ and $q > \frac{2}{n}$, the unique NE is that all users choose DP, i.e., $u_1 = n_1, u_2 = n_2$.*

We end this subsection by proving the existence of NE.

Theorem 9. *For any game instance with two sources, there exists a NE with $u_1 > 0$ and $u_2 = n_2$.*

Proof. By Theorem 5 (4), if $n_1 p \leq q\mu/\phi + n_2 + p$, then the strategy profile that all users choose DP (i.e., $u_1 = n_1, u_2 = n_2$) is a NE. Otherwise, $n_1 p > q\mu/\phi + n_2 + p$. Let $\tilde{m}$ be an integer in interval $[\frac{q\mu/\phi + n_2 + n_1 p - p}{2p}, \frac{q\mu/\phi + n_2 + n_1 p + p}{2p}] = [t_1(n_2) - 1, t_1(n_2)]$, which always admits at least one integer. Note that $n_1 > \frac{q\mu/\phi + n_2 + n_1 p + p}{2p} \geq \tilde{m} > 0$. By Theorem 5 (3), a strategy profile with $u_1 = \tilde{m}$ and $u_2 = n_2$ is a NE. □

3.3 Price of Anarchy

In this section we study the efficiency loss of Nash equilibria, which is measured by the price of anarchy. First, we note that there exists instances with $PoA > 1$, because when $n_2\sqrt{p} + (\sqrt{p} - p)\mu/\phi < n_1 p < q\mu/\phi + n_2 + p$ and $q > \frac{2}{n}$, by Corollary 8 there is a unique NE (i.e., $u_1 = n_1, u_2 = n_2$), however, which is not an optimal solution by Theorem 3. Next, we upper bound the PoA for all game instances. Define $c = \min\{\frac{n_2}{n_1 p}, \frac{n_2 - 1}{n_2 + 1}\}$. The proof is deferred to Appendix.

Theorem 10. *For any instance with two sources, the price of anarchy is $PoA \leq 1 + \frac{\mu/\phi}{n_2 - 2 + c\mu/\phi}$. In particular, when n_2 has a higher-order of μ/ϕ (i.e., $\mu/\phi = o(n_2)$), the PoA approaches to 1.*

4 MULTIPLE SOURCES

In this section, we study the general setting with m sources. Recall that n_i is the number of users arriving at each source $s_i \in \mathcal{S}$, and assume without loss of generality that $n_1 \geq n_2 \geq \cdots \geq n_m$.

Given a strategy profile $\mathbf{p}$, let $u_i = |\{k \in N_i \mid p_{ki} = 1\}|$ be the number of users working with DP (s_i, d), and let $v_i = |\{k \notin N_i \mid p_{ki} = 1\}|$ be the users working with IP through link (s_i, d). Define $y_i = u_i + v_i$ to be the number of users who choose source s_i (including both DP and IP).

For the traffic rate over each link (s_i, d), we can rewrite (1) as

$$T_i(\mathbf{p}) = u_i\phi + v_i p\phi. \tag{16}$$

Each user $k \in N_i$ has a loss rate given by (2), and the total traffic rate of the system is

$$TR(\mathbf{p}) = \sum_{i \in [m]} \left(\frac{\mu T_i}{T_i + \mu}\right) = \mu(m - \sum_{i \in [m]} \frac{\mu}{u_i\phi + v_i p\phi + \mu}). \tag{17}$$

4.1 Computing the optimal solution

We notice that the total traffic rate depends on the number of users working on each source by DP or IP, but not the users' identity. The following lemmas says that $v_i > 0$ only if all users in N_i choose DP.

Lemma 11. *Given $q > 0$, in any optimal solution, for any source s_i, either $u_i = n_i$ or $v_i = 0$ or both hold.*

Proof. Let $\mathbf{p}$ be an optimal solution. Suppose for contradiction that $u_i < n_i, v_i > 0$ for some source s_i. Then there exists a user (say, k) in N_i who chooses IP (say, $(s_i, s_{i'}, d)$ for some $i' \neq i$). Also, since $v_i > 0$, there exist a source $s_j \neq s_i$ and a user $l \in N_j$ who chooses IP (s_j, s_i, d). The total traffic rate is $TR(\mathbf{p}) = \frac{\mu T_i}{T_i + \mu} + \sum_{w \in [m] \setminus \{i\}} \left(\frac{\mu T_w}{T_w + \mu}\right)$.

Now we show that the total traffic rate can be improved by revising $\mathbf{p}$. Let user $k \in N_i$ choose DP, and let user $l \in N_j$ choose IP $(s_j, s_{i'}, d)$. Fixing all others' strategies, denote the new strategy profile by $\mathbf{p}'$, and define u_i', v_i' accordingly. Note that $u_i' = u_i + 1, v_i' = v_i - 1$, and $T_w(\mathbf{p}') = T_w(\mathbf{p})$ for all source $s_w \neq s_i$. Since $q > 0$, we have

$$T_i(\mathbf{p}') = (u_i + 1)\phi + (v_i - 1)p\phi > u_i\phi + v_i p\phi = T_i(\mathbf{p}).$$

Therefore, $TR(\mathbf{p}') > TR(\mathbf{p})$, which contradicts to the optimality. □

Lemma 12. *Given $q > 0$, in any optimal solution, there must exist $\tilde{i} \in [m]$, such that $v_l = 0$ for all $l \leq \tilde{i}$, and $u_j = n_j$ for all $j > \tilde{i}$.*

Proof. Given an optimal solution $\mathbf{p}$, suppose for contradiction that there exist $i, j \in [m]$ $(i < j)$ such that $v_i > 0$ and $u_j < n_j$. By Lemma 11, we have $u_i = n_i$ and $v_j = 0$. There exists a source $s_{i'}$ and a user $k \in N_{i'}$ selecting IP $(s_{i'}, s_i, d)$. There exists a source $s_{j'}$ $(j' \neq j)$ and a user $k' \in N_j$ selecting IP $(s_j, s_{j'}, d)$. Note that when $i' = j$ and $j' = i$, users k and k' may coincide. The total traffic rate is

$$\begin{aligned} TR(\mathbf{p}) &= \frac{\mu T_i}{T_i + \mu} + \frac{\mu T_j}{T_j + \mu} + \sum_{w \in [m] \setminus \{i,j\}} \left(\frac{\mu T_w}{T_w + \mu} \right) \\ &= \mu\left(2 - \frac{1}{T_i + \mu} - \frac{1}{T_j + \mu}\right) + \sum_{w \in [m] \setminus \{i,j\}} \left(\frac{\mu T_w}{T_w + \mu} \right). \end{aligned}$$

Now we show that the total traffic rate can be improved by revising $\mathbf{p}$. Let user k choose IP $(s_{i'}, s_{j'}, d)$ if $i' \neq j'$ and choose DP $(s_{i'}, d)$ if $i' = j'$. Let user $k' \in N_j$ choose DP. Fixing all others' strategies, denote the new strategy profile by $\mathbf{p}'$, and define u_i', v_i' accordingly. Note that $v_i' = v_i - 1, u_j' = u_j + 1$, and $T_w(\mathbf{p}') = T_w(\mathbf{p})$ for all other source $s_w \neq s_i, s_j$. Since $i < j$, it follows that $n_i \geq n_j > u_j$. Therefore, we have

$$\begin{aligned} \frac{1}{T_i + \mu} + \frac{1}{T_j + \mu} &= \frac{1}{n_i\phi + v_i p\phi + \mu} + \frac{1}{u_j\phi + \mu} \\ &> \frac{1}{n_i\phi + v_i' p\phi + \mu} + \frac{1}{u_j'\phi + \mu} \\ &= \frac{1}{T_i' + \mu} + \frac{1}{T_j' + \mu}, \end{aligned}$$

which indicates that $TR(\mathbf{p}) < TR(\mathbf{p}')$, a contradiction. □

Now we are ready to present the polynomial-time algorithm for computing an optimal solution. The main idea is searching for $\tilde{i}$ in Lemma 12. For every possible candidate of $\tilde{i}$, let B be the number of users selecting IP, all of whom come from $L = \{1, \ldots, \tilde{i}\}$, and go to $R = \{\tilde{i} + 1, \ldots, m\}$. For every possible value of B, we compute the best possible approach for extracting the B users from L and distributing them over R.

Algorithm 1 *ALG* **for computing optimal solution.**

Input: m source with $n_1 \geq n_2 \geq \cdots \geq n_m$, ϕ, μ, q
Output: $(u_i^*, v_i^*)_{i \in N}$

```
Initialize u_i = v_i = 0 for all i ∈ [m]. TR* = 0
for ĩ = 1, ..., m do
    Let v_l = 0 for all l = 1, 2, ..., ĩ
    Let u_j = n_j for all j = ĩ + 1, ..., m
    for B = 1, 2, ..., Σ_{l=1}^{ĩ} n_l do
        (a) Compute (u_l)_{l∈[ĩ]} so that Σ_{l=1}^{ĩ} (n_l − u_l) = B and the
            values of u_l are as even as possible.
        (b) Compute (v_j)_{j>ĩ} so that Σ_{j=ĩ+1}^{m} v_j = B and the values
            of n_j φ + v_j p φ + μ are as even as possible.
        (c) Compute TR with respect to (u_i, v_i)_{i∈N}
        if TR > TR* then
            TR* ← TR, and (u_i*, v_i*)_{i∈N} ← (u_i, v_i)_{i∈N}
        end if
    end for
end for
```

In the above algorithm, the goal of Step (a) is to make T_l (and thus $\frac{1}{T_l + \mu}$) as even as possible, and it can be realized by initializing $u_l = n_l$, and then removing agents one by one from the highest u_l and updating, until B agents have been removed. The goal of Step (b) is to make T_j (and thus $\frac{1}{T_j + \mu}$) as even as possible, and can be realized by initializing $v_j = 0$, and then adding agents one by one to $v_{j'} = \arg\min_{j > \tilde{i}} n_j\phi + v_j p\phi + \mu$ and updating, until B agents have been added.

Though the output of the algorithm is $(u_i^*, v_i^*)_{i \in N}$, we can easily extend it to a corresponding strategy profile because the situation for each source s_i has been determined. Next we prove that it achieves the system optimum.

Theorem 13. *Algorithm 1 computes an optimal solution in $O(mn^2)$ time.*

Proof. We first show the correctness. By Lemma 12, there exists an index $\tilde{i}$ to divide the sources into two sets $L = \{s_l \mid l \leq \tilde{i}\}$ and $R = \{s_j \mid j > \tilde{i}\}$. In the first loop of our algorithm, we traverse all indexes in $[m]$. In the second loop, we traverse all possible number of users who select IP, and given any such a number B, we compute the best possible approach to extract the B users from L and distribute them over R. So all possible optimal solutions have been searched by the algorithm, which gives the optimality.

For the time complexity, we have m iterations in the first loop, at most n iterations in the second loop, and the time for each iteration is $O(n)$. So the running time is $O(mn^2)$. □

Intuitively, when the transmission loss probability is sufficiently large, all packets should go through DP, and when there is no transmission loss, the load of packets should be distributed evenly over all sources. We verify the intuition as follows.

Corollary 14. *If $q = 1$, the unique optimal solution is that all users choose DP (i.e., $u_i = n_i, v_i = 0, \forall i \in M$). If $q = 0$, a strategy profile $\mathbf{p}$ is optimal if and only if $|y_i - y_j| \leq 1$ for all $i, j \in [m]$.*

Proof. If $q = 1$, $TR = \sum_{i\in[m]} \frac{\mu u_i\phi}{u_i\phi+\mu}$ is increasing with respect to every u_i. By the monotonicity, the optimum is achieved when $u_i = n_i$.

If $q = 0$, suppose for contradiction that there exist $i, j \in [m]$ in an optimal solution $\mathbf{p}$ such that $y_i - y_j \geq 2$. The total traffic rate is $TR = \mu(m - \sum_{k\in[m]} \frac{\mu}{y_k\phi+\mu})$. Consider a new strategy profile $\mathbf{p}'$ with $y_i' = y_i - 1, y_j' = y_j + 1$, i.e., a user who chooses source s_i deviates to s_j. Then the total traffic rate becomes $TR' = \mu(m - \sum_{k\in[m]\setminus\{i,j\}} \frac{\mu}{y_k\phi+\mu} - \frac{\mu}{y_i'\phi+\mu} - \frac{\mu}{y_j'\phi+\mu}) > TR$, a contradiction with the optimality. □

4.2 Characterization of NEs

Informally, a NE should satisfy that, for an user who selects DP, she will not deviate to any IP; for an user who selects IP, she will not deviate to DP nor another IP. We formalize it as the following characterization.

Theorem 15. *Given an arbitrary strategy profile $\mathbf{p}$ with $(u_i, v_i)_{i\in[m]}$, let $i^* \in \arg\min_{i\in[m]}\{u_i + v_i p\}$, and let $x_{ij} \in \{0, 1\}$ indicate if there is some user selecting IP (s_i, s_j, d). Then, $\mathbf{p}$ is a NE, if and only if the following conditions are satisfied:*

- *for all $i \in [m]$ with $u_i > 0$, we have*

$$p(u_i + v_i p) \leq u_{i^*} + v_{i^*} p + p + \frac{q\mu}{\phi}; \tag{18}$$

- *for all $i \in [m]$, all $l \neq i$ with $x_{il} = 1$, we have*

$$u_l + v_l p \leq p(u_i + 1 + v_i p) - \frac{q\mu}{\phi}, \tag{19}$$

and

$$u_l + v_l p \leq u_{i^*} + v_{i^*} p + p. \tag{20}$$

Proof. Suppose $\mathbf{p}$ is a NE. By Definition 1, for any source $s_i \in S$ and user $k \in N_i$, if k selects DP, then for any $j \neq i$ and strategy $\mathbf{p}_k'$ with $p_{kj} = 1$, it should have $LR_k(\mathbf{p}_k, \mathbf{p}_{-k}) \leq LR_k(\mathbf{p}_k', \mathbf{p}_{-k})$. It is equivalent to

$$\begin{aligned}
&1 - \frac{\mu}{u_i\phi + v_i p\phi + \mu} \leq q + p \cdot \left(1 - \frac{\mu}{u_j\phi + (v_j + 1)p\phi + \mu}\right) \\
\Leftrightarrow\ & \frac{p}{u_j\phi + (v_j + 1)p\phi + \mu} \leq \frac{1}{u_i\phi + v_i p\phi + \mu} \\
\Leftrightarrow\ & p(u_i + v_i p) - \frac{q\mu}{\phi} \leq u_j + (v_j + 1)p,
\end{aligned}$$

as given by Equation (18).

If k selects IP $(s_i, s_l, 0)$ for some $l \neq i$, then she will not deviate to DP $(s_i, 0)$, that is, $LR_k(\mathbf{p}_k, \mathbf{p}_{-k}) \leq LR_k(\mathbf{p}_k', \mathbf{p}_{-k})$, where $\mathbf{p}_k'$ is a strategy with $p_{ki=1}$. It is equivalent to

$$\begin{aligned}
&q + p \cdot \left(1 - \frac{\mu}{u_l\phi + v_l p\phi + \mu}\right) \leq 1 - \frac{\mu}{(u_i + 1)\phi + v_i p\phi + \mu} \\
\Leftrightarrow\ & \frac{1}{(u_i + 1)\phi + v_i p\phi + \mu} \leq \frac{p}{u_l\phi + v_l p\phi + \mu} \\
\Leftrightarrow\ & u_l + v_l p \leq p(u_i + 1 + v_i p) - \frac{q\mu}{\phi},
\end{aligned}$$

as given by Equation (19). Moreover, she will not deviate to any other IP $(s_i, s_j, 0)$ with $j \neq i, l$, that is

$$\begin{aligned}
&1 - \frac{\mu}{u_l\phi + v_l p\phi + \mu} \leq 1 - \frac{\mu}{u_j\phi + (v_j + 1)p\phi + \mu} \\
\Longleftrightarrow\ & u_l + v_l p \leq u_j + (v_j + 1)p,
\end{aligned}$$

as given by Equation (20). □

4.3 Price of Anarchy

We investigate the price of anarchy in this section. In the following two lemmas, we first give an upper bound on the optimal total traffic rate, and then give a lower bound on the total traffic rate of any NE. Combining these two bounds, we derive the PoA.

Lemma 16. *In any optimal solution $\mathbf{p}$, for any $i \in [m]$, it must be $u_i + v_i p \leq n_1$. In addition, $TR(opt) \leq \mu(m - \frac{m\mu}{n_1\phi+\mu})$.*

Proof. First, for $i = 1$, by Lemma 12, we have $v_1 = 0$, and thus it satisfies $u_1+v_1 p = u_1 \leq n_1$. For any $i > 1$, suppose for contradiction that $u_i + v_i p > n_1$. Then $v_i > 0$, and there exists a source s_j and a user $k \in N_j$ who works with s_i via IP (s_j, s_i, d). By Lemma 11, it must be $v_j = 0$, and thus $T_j = u_j\phi$ The total traffic rate is

$$TR(\mathbf{p}) = \mu\left(m - \frac{\mu}{T_j + \mu} - \frac{\mu}{T_i + \mu} - \sum_{w\in[m]\setminus\{i,j\}} \frac{\mu}{T_w + \mu}\right).$$

We show that the total traffic rate can be improved by letting user k choose DP (s_j, d). Fixing the strategies of all others, denote by $\mathbf{p}'$ the new strategy profile, and define $(u_w', v_w', T_w')_{w\in[m]}$ accordingly. Note that $u_j' = u_j + 1$, $v_i' = v_i - 1$, $u_i' = u_i$, and $T_w' = T_w$ for any $w \in [m]\setminus\{i, j\}$. Since $u_i + v_i p > n_1 \geq n_j \geq u_j$, we have

$$\begin{aligned}
\frac{1}{T_j + \mu} + \frac{1}{T_i + \mu} &= \frac{1}{u_j\phi + \mu} + \frac{1}{u_i\phi + v_i p\phi + \mu} \\
&> \frac{1}{u_j'\phi + \mu} + \frac{1}{u_i'\phi + v_i' p\phi + \mu} \\
&= \frac{1}{T_j' + \mu} + \frac{1}{T_i' + \mu}.
\end{aligned}$$

It indicates that $TR(\mathbf{p}') > TR(\mathbf{p})$, a contradiction. □

Lemma 17. *Let $z = \min\{n_m, \frac{n}{4m} - p - \frac{q\mu}{\phi}\}$. For every NE $\mathbf{p}$, the total traffic rate is at least $TR(\mathbf{p}) \geq \mu(m - \frac{m\mu}{z\phi+\mu})$.*

Proof. Let $i^* = \arg\min_{i\in[m]}\{u_i + v_i p\}$. Since $TR(\mathbf{p}) \geq \mu(m - \frac{m\mu}{(u_{i^*}+v_{i^*}p)\phi+\mu})$, it suffices to prove that $u_{i^*} + v_{i^*} p \geq z$. If $u_{i^*} + v_{i^*} p \geq n_m$, it is done. We only need to consider the case when $u_{i^*} + v_{i^*} p < n_m \leq n_{i^*}$. There exists some users in N_{i^*} selecting IP. By Theorem 15 (19), we have

$$u_{i^*} + v_{i^*} p \leq p(u_{i^*} + 1 + v_{i^*} p) - \frac{q\mu}{\phi},$$

which indicates $p \geq 0.5$.

By Theorem 15, for each $i \in [m]$, if $u_i > 0$, then $p(u_i + v_i p) \leq u_{i^*} + v_{i^*} p + p + \frac{q\mu}{\phi}$; if $v_i > 0$, then $u_i + v_i p \leq u_{i^*} + v_{i^*} p + p$. In both cases, we obtain $u_i + v_i/2 \leq 2(u_{i^*} + v_{i^*} p + p + \frac{q\mu}{\phi})$. Summing up over all $i \in [m]$, we have

$$n/2 \leq \sum_{i\in[m]} (u_i + v_i/2) \leq 2m(u_{i^*} + v_{i^*} p + p + \frac{q\mu}{\phi}),$$

which implies that $u_{i^*} + v_{i^*} p \geq \frac{n}{4m} - p - \frac{q\mu}{\phi}$. □

Now we are ready to present the results on PoA.

Theorem 18. *For any instance with m sources, the price of anarchy is* $PoA \leq 1 + \frac{n_1\mu}{n_1 z\phi + z\mu}$.

Proof. Combining the upper bound on $TR(opt)$ in Lemma 16 and the lower bound on $TR(\mathbf{p})$ for any NE $\mathbf{p}$ in Lemma 17, it follows

$$PoA \leq \frac{m - \frac{m\mu}{n_1\phi+\mu}}{m - \frac{m\mu}{z\phi+\mu}} = \frac{n_1(z\phi + \mu)}{z(n_1\phi + \mu)} = 1 + \frac{n_1\mu}{n_1 z\phi + z\mu}.$$

□

5 NUMERICAL EXPERIMENTS

Through numerical simulations, we explore the impact of traffic condition on network performance, i.e., the total traffic rate and PoA. Recall that the traffic flow originating from each user is Poisson with rate ϕ, the service rate of each direct link is μ, and the loss probability over each relay link is q. Assume $\phi = 1$ by normalization.

As shown in Figure 2, we conduct experiments with two sources, for a range of q, μ, n_1. The optimal solution and all NEs can be found efficiently by Theorems 3 and 5. In all these experiments, the total traffics in the optimal solution and in a worst NE are very close, resulting in a PoA of less than 1.08. Particularly, in Figure 2a we consider a range of q, fixing $n_1 = 1000, n_2 = 100, \mu = 300$. When the loss probability q over relay link (s_1, s_2) increases, the total traffics in both optimal solution and NE decrease, and the PoA is first increasing then decreasing. An intuitive explanation for the decreasing is, when q is large enough, in both optimal solution and NE, no player prefers IP, and the solution that all users select DP is not only optimal but also stable.

Figure 2b plots the performances for a range of service rate μ, fixing $n_1 = 1000, n_2 = 100, q = 0.3$. When μ increases, the PoA approaches to 1, and both traffic rate curves increase, because the improved service clears collision and congestion. Figure 2c plots the performances for a range of n_1, fixing $n_2 = 100, q = 0.7, \mu = 300$. Also, the PoA are first increasing and then decreasing, and it tends to 1 when n_1 is large enough.

We also conduct experiments for multiple sources, see Figure 3. While the optimal solution can be easily computed by Algorithm 1, it is harder to find all NEs, even given Theorem 15. Thus we consider small m and n (i.e., $m = 3$, $n_1 = 12$, $n_2 = 8$, $n_3 = 4$) in Figures 3a and 3b. In all these experiments, the PoA is less than 1.02. Figure 3a plots the performances for a range of q, fixing $\mu = 1$. When $q = 0$ and $q = 1$, the PoA is exactly 1. Figure 3b plots the performances for a range of μ, fixing $q = 0.5$. The PoA converges to 1 when μ goes to infinity, because when the service rate is large enough compared with arrival rate, there will be a sufficiently small congestion loss and all users would like to choose DP. Figure 3c plots the performances for a range of n_1, fixing $n_2 = 4$ and $n_3 = 2$.

Besides the selected settings and performances shown in Figures 2 and 3, indeed we have implemented a large amount experiments for two/multi-source cases. In most cases, the total traffics in optimum and NE are too close to distinguish from the images. The PoA is less than 1.1 in all the experiments. Moreover, we observe that the number of NEs changes w.r.t. q: when $q \leq 0.1$, there are often multiple NEs appears; when q is larger, the NE is usually unique.

6 CONCLUSION

In this work, we investigated the load balancing game in loss networks, in which self-interested users want to minimize the loss probability of their packets in the transmission process by selecting suitable routing. We proposed an efficient algorithm for maximizing the total traffic rate, and provided a characterization of (pure-strategy) Nash equilibria, in which no user can decrease his/her loss probability by unilateral deviation. We provided theoretical guarantees on the PoA and conducted numerical experiments, both showing that the efficiency loss due to selfish behaviors is small.

There are many future directions that are worth exploring. First, we only focus on pure strategies of players in this work, and an immediate and natural question is how the users act when mixed strategies are allowed. Second, the servers (source nodes) in our setting is homogeneous with the same service rate μ, and it would be interesting to investigate heterogeneous servers where each s_i serves a different purpose or has a different service rate μ_i. Moreover, while we only consider direct path and one-hop indirect paths, a more general scenario where players can choose multi-hop indirect paths to destination can be taken into consideration.

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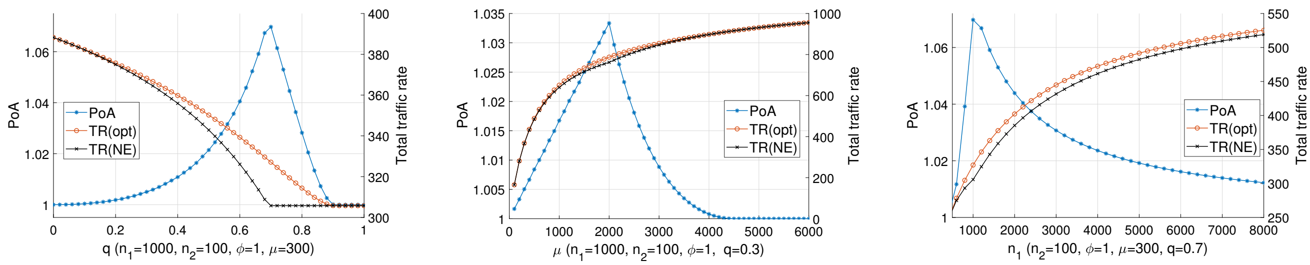
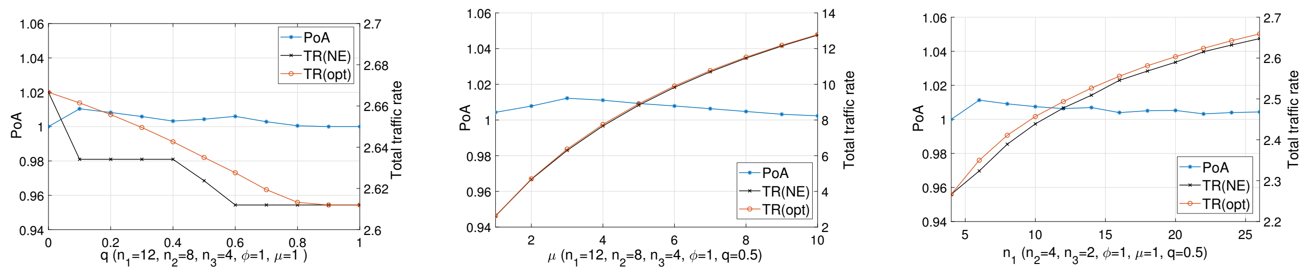


(a) (b) (c)

Figure 2: The PoA and total traffics in opt and NE, with two sources. (a) compares a range of $q \in [0, 1]$; (b) compares a range of $\mu \in [1, 6000]$; (c) compares a range of $n_1 \in [500, 8000]$.



(a) (b) (c)

Figure 3: The PoA and total traffics in opt and NE, with three sources. (a) compares a range of $q \in [0, 1]$; (b) compares a range of $\mu \in [1, 10]$; (c) compares a range of $n_1 \in [4, 26]$.

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APPENDIX

Proof of Theorem 10

Proof. Given any game instance, let opt be an optimal solution. By Eq. (7), the optimal total traffic rate is upper bounded by

$$\begin{aligned} TR(opt) &= \mu\left(\frac{T_1(opt)}{T_1(opt)+\mu} - \frac{T_2(opt)}{T_2(opt)+\mu}\right) \\ &\le \mu\left(\frac{n_1}{n_1+\mu/\phi} + \frac{n_2+n_1p}{n_2+n_1p+\mu/\phi}\right). \end{aligned}$$

Let $\mathbf{p}$ be an arbitrary NE. If $\mathbf{p}$ falls in Case 2 of Theorem 5, the traffic rate is bounded by

$$\begin{aligned} TR(\mathbf{p}) &= \mu\left(2 - \frac{\mu/\phi}{u_1+(n_2-u_2)p+\mu/\phi} - \frac{\mu/\phi}{u_2+(n_1-u_1)p+\mu/\phi}\right) \\ &\ge \mu\left(2 - \frac{\mu/\phi}{\frac{n_1-1}{2}+\frac{n_2p}{2}+\mu/\phi} - \frac{\mu/\phi}{u_2+(n_1-u_1)p+\mu/\phi}\right) \\ &\ge \mu\left(2 - \frac{\mu/\phi}{\frac{n_1-1}{2}+\mu/\phi} - \frac{\mu/\phi}{\frac{n_2+n_1p-1}{2}+\mu/\phi}\right) \\ &= \mu\left(\frac{n_1-1}{n_1-1+2\mu/\phi} + \frac{n_2+n_1p-1}{n_2+n_1p-1+2\mu/\phi}\right), \end{aligned}$$

where the first inequality comes from $u_1 \ge t_1(u_2) - 1$, and the second inequality comes from $u_2 \ge t_2(u_1) - 1$. After calculation and relaxation, we have

$$\frac{TR(opt)}{TR(\mathbf{p})} \le 1 + \frac{\mu/\phi}{n_2 - 2 + \frac{n_2-1}{n_2+1}\mu/\phi}.$$

If $\mathbf{p}$ falls in Case 3, the traffic rate is bounded by

$$\begin{aligned} TR(\mathbf{p}) &= \mu\left(2 - \frac{\mu/\phi}{u_1+\mu/\phi} - \frac{\mu/\phi}{n_2+(n_1-u_1)p+\mu/\phi}\right) \\ &\ge \mu\left(2 - \frac{\mu/\phi}{\frac{n_1-1}{2}+\frac{n_2}{2p}+\mu/\phi} - \frac{\mu/\phi}{n_2+\mu/\phi}\right) \\ &= \mu\left(\frac{n_1-1}{n_1-1+2\mu/\phi} + \frac{n_2}{n_2+\mu/\phi}\right), \end{aligned}$$

where the first inequality comes from $u_1 \geq t_1(n_2) - 1$, and the second inequality comes from $u_1 \leq t_1(u_2)$. After calculation and relaxation, we have

$$\frac{TR(opt)}{TR(\mathbf{p})} \leq \max\{1 + \frac{\mu/\phi}{n_2 + \frac{n_2}{n_1 p}\mu/\phi}, 1 + \frac{\mu/\phi}{n_1 - 2 + \frac{n_1-1}{n_1+1}\mu/\phi}\}.$$

If $\mathbf{p}$ falls in Case 4, the traffic rate is

$$TR(\mathbf{p}) = \mu\left(\frac{n_1}{n_1 + \mu/\phi} + \frac{n_2}{n_2 + \mu/\phi}\right).$$

By a similar calculation, we have

$$\frac{TR(opt)}{TR(\mathbf{p})} \leq 1 + \frac{\mu/\phi}{n_2 + \frac{n_2}{n_1 p}\mu/\phi}.$$

Combining the above three cases, it completes the proof □