Accomplice Manipulation of the Deferred Acceptance Algorithm

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ABSTRACT

The deferred acceptance algorithm is an elegant solution to the stable matching problem that guarantees optimality and truthfulness for one side of the market. Despite these desirable guarantees, it is susceptible to strategic misreporting of preferences by the agents on the other side. We study a novel model of strategic behavior under the deferred acceptance algorithm: manipulation through an *accomplice*. Here, an agent on the proposed-to side (say, a woman) partners with an agent on the proposing side—an accomplice—to manipulate on her behalf (possibly at the expense of worsening his match). We show that the optimal manipulation strategy for an accomplice comprises of promoting exactly one woman in his true list (i.e., an *inconspicuous* manipulation). This structural result immediately gives a polynomial-time algorithm for computing an optimal accomplice manipulation. We also study the conditions under which the manipulated matching is stable with respect to the true preferences. Our experimental results show that accomplice manipulation outperforms self manipulation both in terms of the frequency of occurrence as well as the quality of matched partners.

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1 INTRODUCTION

The deferred acceptance (DA) algorithm [9] is a crowning achievement of the theory of two-sided matching [18], and forms the backbone of a wide array of real-world matching markets such as entry-level labor markets [21, 24] and school choice [1, 2]. Under this algorithm, one side of the market (colloquially, the *men*) makes proposals to the other side (the *women*) subject to either immediate rejection or tentative acceptance. A key property of the DA algorithm is *stability* which says that no pair of unmatched agents should prefer each other over their assigned partners. This property has played a significant role in the long-term success of several real-world matching markets [22, 23].

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The attractive stability guarantee of the DA algorithm, however, comes at the cost of incentives, as any stable matching procedure is known to be vulnerable to strategic misreporting of preferences [20]. The special proposal-rejection structure of the DA algorithm makes truth-telling a dominant strategy for the proposing side, i.e., the men [8, 20], implying that any strategic behavior must occur on the proposed-to side, i.e., the women. This model of strategic behavior by a woman—which we call *self manipulation*—has been the subject of extensive study in economics and computer science [6–8, 11, 16, 17, 26, 27].

Our interest in this work is in studying a different model of strategic behavior under the DA algorithm called *manipulation through an accomplice* [4]. Under this model, a woman reports her preferences truthfully, but asks an agent on the proposing side (a.k.a. an accomplice) to manipulate the outcome on her behalf, possibly worsening his match in the process.

Such a strategic alliance can naturally arise in the assignment of students to schools under a school-proposing setup, where a "well-connected" student could have a school administrator manipulate on his/her behalf, possibly at a small loss to the school. Similarly, in a student-proposing setting, schools can strategize by making themselves appear less attractive to students from low-income backgrounds (say, by increasing rent of dormitories or requiring students to purchase expensive uniforms), thus forcing a change in the students' preferences [12]. Yet another justification for accomplice manipulation comes from thinking about strategic behaviour as a *control* problem, wherein a woman can bribe a man to lie on her behalf. Bribery has been extensively studied in computational social choice in the context of voting, and our work can be seen as investigating this phenomenon in the two-sided matching framework.

At first glance, manipulation through an accomplice might not seem any more powerful than self manipulation, as the latter provides direct control over the preferences of the manipulator. Interestingly, there exist instances where this intuition turns out to be wrong.

Example 1.1 (Accomplice vs. self). Consider the following preference profile where the DA outcome is underlined. The notation "$m_1 : w_3\ w_2\ w_1\ w_4$" denotes that for man m_1, the first choice woman is w_3, the second choice is w_2, and so on.

$\boldsymbol{m_1}$:	$\underline{w_3^*}$	w_2	w_1	w_4	$\boldsymbol{w_1}$:	m_4	m_3^*	m_1	$\underline{m_2}$
m_2:	$\underline{w_1}$	w_4^*	w_2	w_3	w_2:	$\underline{m_4^*}$	m_3	m_2	m_1
m_3:	w_2	$\underline{w_4}$	w_1^*	w_3	w_3:	m_3	$\underline{m_1^*}$	m_2	m_4
m_4:	$\underline{w_2^*}$	w_1	w_3	w_4	w_4:	m_2^*	m_1	$\underline{m_3}$	m_4

Suppose w_1 seeks to improve her match via manipulation. The optimal self manipulation strategy for w_1 is truth-telling, as m_2 is the only man who proposes to her under the DA algorithm. On the other

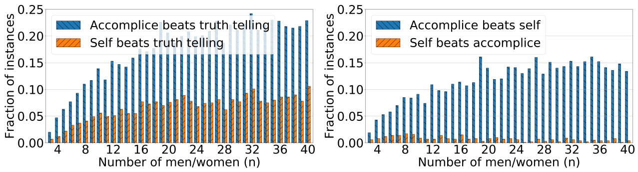

Figure 1: Comparing no-regret accomplice manipulation and self manipulation against truthful reporting (top) and against each other (bottom).

hand, w_1 can improve her outcome by asking m_1 to misreport on her behalf. Indeed, if m_1 misreports by declaring $\succ'_{m_1} := w_1 \succ w_3 \succ w_2 \succ w_4$, then w_1's match improves from m_2 to m_3 (the new DA matching is marked by $*$). Notice that the accomplice m_1 preserves his initial match, meaning he does not incur any 'regret'. □

The above example highlights that accomplice manipulation could, in principle, have an advantage over self manipulation. However, it is not a priori clear how *frequent* such an advantage might be in a typical matching scenario. To investigate the latter question, we take a quick experimental detour.

Accomplice manipulation is a viable strategic behavior. We simulate a two-sided matching scenario for an increasingly larger set of agents (specifically, $n \in \{3, \ldots, 40\}$, where n is the number of men/women) and for each setting, generate 1000 preference profiles uniformly at random. For each profile, we compute the optimal self manipulation under the DA algorithm for a fixed woman [26], as well as the optimal accomplice manipulation by any man (we allow any man to be chosen as an accomplice as long as he is not worse off, i.e., a *no-regret* accomplice manipulation). Figure 1 illustrates the fraction of instances where accomplice and self manipulation are strictly more beneficial than truthful reporting, as well as how they compare against each other.

Example 1.1 and Figure 1 suggest that the incentive for manipulation through an accomplice is not only present but actually more prevalent than self manipulation. Additionally, as we discuss later in our experimental results, women are expected to receive *better* matches when manipulating through an accomplice (Figure 2). These promising observations call for a systematic study of the structural and computational aspects of the accomplice manipulation problem, which is the focus of our work.

Our contributions. We consider two models of strategic behavior—*no-regret* manipulation (wherein the accomplice's own match doesn't worsen upon misreporting) and *with-regret* manipulation (where the accomplice could get a worse match)—and make the following contributions:

- **No-regret manipulation**: Our main theoretical result (Theorem 4.3) is that any optimal no-regret accomplice manipulation can be simulated by promoting exactly one woman in the true preference list of the accomplice; in other words, the manipulation is *inconspicuous* [27]. This structural finding immediately gives a polynomial-time algorithm for computing an optimal manipulation (Corollary 2). We also show that the inconspicuous no-regret strategy results in a matching that is stable with respect to the true preferences (Corollary 3).
- **With-regret manipulation**: For the more permissible strategy space that allows the accomplice to incur regret, the optimal manipulation strategy once again turns out be inconspicuous (Theorem 5.2). However, in contrast to the no-regret case, the inconspicuous with-regret strategy is no longer guaranteed to be stability-preserving (Example 5.1). Nevertheless, any blocking pair can be shown to necessarily involve the accomplice (Proposition 2). This property justifies the use of an accomplice who can be encouraged to tolerate some regret to benefit a woman.
- **Experiments**: On the experimental front, we work with preferences generated uniformly at random, and find that accomplice manipulation outperforms self manipulation with respect to the frequency of occurrence, the quality of matched partners, and the fraction of women who can improve their matches (Section 6).

2 RELATED WORK

The impossibility result of Roth [20] on the conflict between strategyproofness and stability has led to extensive follow-up research. Much of the earlier work in this direction focused on *truncation* manipulation [5, 10, 14, 25], where the misreported preference list is required to be a prefix of the true list. In the context of accomplice manipulation, however, the truncation manipulation problem becomes trivial. Indeed, for any fixed accomplice, it is easy to see that a truncation strategy is never better than truthful reporting as the set of proposals under the former is always a subset of those under the latter.

The literature on self manipulation via *permutation* is more recent and has focused on computational aspects. Teo et al. [26] provided a polynomial-time algorithm for computing the optimal permutation manipulation by a woman under the men-proposing DA algorithm. Vaish and Garg [27] showed that an optimal permutation manipulation is, without loss of generality, inconspicuous. They also studied conditions under which the manipulated outcome is stable with respect to the true preferences. Deng et al. [7] studied permutation manipulation by a coalition of women.

Huang [13] has studied (weakly) Pareto improving permutation manipulation by a coalition of men, revisiting the result of Dubins and Freedman [8] on the impossibility of manipulations that are strictly improving for every member of the coalition.

The accomplice manipulation model was proposed by Bendlin and Hosseini [4], who noted that manipulation through an accomplice can be strictly more preferable for the woman than optimal self manipulation. However, they left the structural and computational questions open.

Balinski and Sönmez [3] studied a closely related problem in school choice wherein the students have an incentive to perform worse on tests, or make themselves appear less preferable to colleges when the college-optimal algorithm is used. Hatfield et al. [12] similarly showed that in a student-optimal mechanism, schools have the incentive to deliberately make themselves look less attractive to "undesirable" students. For example, a private school that is legally required to cap its tuition fee for low-income students could make itself less attractive by increasing the rent in dormitories or requiring the students to purchase expensive uniforms.

3 PRELIMINARIES

3.1 Stable Matching Problem

Problem setup. An instance of the *stable marriage problem* [9] is specified by the tuple $\langle M, W, \succ \rangle$, where M is a set of n men, W is a set of n women, and $\succ$ is a *preference profile* which consists of the preference lists of all agents. The preference list of any man $m \in M$, denoted by $\succ_m$, is a strict total order over all women in W (for any $w \in W$, the list $\succ_w$ is defined analogously).

We use the shorthand $w_1 \succeq_m w_2$ to denote 'either $w_1 \succ_m w_2$ or $w_1 =_m w_2$' (the latter relation denotes that man m is indifferent between w_1 and w_2), and write $\succ_{-m}$ to denote the preference lists of all men and women except man m; thus, $\succ = \{\succ_{-m}, \succ_m\}$.

Stable matchings. A *matching* is a function $\mu : M \cup W \rightarrow M \cup W$ such that $\mu(m) \in W$ for all $m \in M$, $\mu(w) \in M$ for all $w \in W$, and $\mu(m) = w$ if and only if $\mu(w) = m$. A matching μ admits a *blocking pair* with respect to the preference profile $\succ$ if there is a man-woman pair (m, w) who prefer each other over their assigned partners under μ, i.e., $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A *stable matching* is one that does not admit any blocking pair. We will write $S_\succ$ to denote the set of all matchings that are stable with respect to $\succ$. In addition, for any pair of matchings μ, μ', we will write $\mu \succeq_M \mu'$ to denote $\mu(m) \succeq_m \mu'(m)$ for all $m \in M$ (and $\mu \succeq_W \mu'$ for the women).

Deferred acceptance algorithm. Given a preference profile $\succ$, the Deferred Acceptance (DA) algorithm of Gale and Shapley [9] proceeds in rounds. In each round, the algorithm consists of a *proposal* phase, where each man who is currently unmatched proposes to his favorite woman from among those who have not rejected him yet, followed by a *rejection* phase where each woman tentatively accepts her favorite proposal and rejects the rest. The algorithm terminates when no further proposals can be made. Gale and Shapley [9] showed that given any profile $\succ$ as input, the DA algorithm always returns a stable matching as output; we denote this matching by $\mathrm{DA}(\succ)$. They also observed that this matching is *men-optimal*, i.e., it assigns each man his favorite stable partner among all stable matchings in $S_\succ$. McVitie and Wilson [19] subsequently showed that this matching is also *women-pessimal*.

PROPOSITION 1 ([9, 19]). *Let $\succ$ be a preference profile and let $\mu := \mathrm{DA}(\succ)$. Then, $\mu \in S_\succ$. Furthermore, for any $\mu' \in S_\succ$, $\mu \succeq_M \mu'$ and $\mu' \succeq_W \mu$.*

Accomplice manipulation. Under this model of strategic behavior, a woman w, instead of misreporting herself, has a man m provide a manipulated preference list, say $\succ'_m$, in order to improve her match. Given a preference profile $\succ$, we say that w can *manipulate through accomplice m* if $\mu'(w) \succ_w \mu(w)$, where $\mu := \mathrm{DA}(\succ)$, $\succ' := \{\succ_{-m}, \succ'_m\}$, and $\mu' := \mathrm{DA}(\succ')$. We will often refer to (m, w) as the *manipulating pair* (not to be confused with a blocking pair).

Throughout this paper, any manipulation will be assumed to be *optimal* unless stated otherwise. That is, there exists no other list $\succ''_m$ for the accomplice m such that $\mu''(w) \succ_w \mu'(w)$, where $\succ'' := \{\succ_{-m}, \succ''_m\}$, and $\mu'' := \mathrm{DA}(\succ'')$. Note that we assume that the manipulator has *full information* about the preferences of other agents. Extending our results to incomplete or uncertain information settings is an interesting direction for future research.

No-regret and with-regret manipulation. We say that the accomplice m incurs *regret* if his match worsens upon misreporting, i.e., $\mu(m) \succ_m \mu'(m)$. It is known that the DA algorithm is strategyproof for the proposing side [8], which means that no man can improve his match by unilaterally misreporting his preferences. Therefore, for any man $m \in M$ and for any misreport $\succ'_m$, we have that $\mu(m) \succeq_m \mu'(m)$. Thus, equivalently, we say that man m incurs regret if $\mu(m) \neq \mu'(m)$.

We will consider two models of accomplice manipulation in this paper: *no-regret manipulation* wherein only those misreports $\succ'_m$ are allowed under which $\mu(m) = \mu'(m)$, and *with-regret manipulation*, where the accomplice is allowed (but not required) to incur regret. Thus, any no-regret strategy is also a with-regret strategy. Recall that the misreport in Example 1.1 was a no-regret manipulation.

Stability relaxations. For any preference profile $\succ$ and a fixed man $m \in M$, we say that a matching μ is *m-stable* [4] with respect to $\succ$ if any blocking pair (if one exists) in μ involves the man m. That is, for any pair (m', w') that blocks μ under $\succ$, we have $m' = m$. Clearly, a stable matching is also m-stable. Under accomplice manipulation, it can be shown that any matching μ' that is stable with respect to the manipulated profile (in particular, when $\mu' = \mathrm{DA}(\succ')$) is m-stable with respect to the true profile $\succ$ (Proposition 2). We note that Proposition 2 strengthens a result of Bendlin and Hosseini [4] who proved a similar statement only for a DA matching. The proof of this result, along with all other omitted proofs, can be found in the appendix.

PROPOSITION 2. *Let $\succ$ denote the true preference profile. For any man m, let $\succ' := \{\succ_{-m}, \succ'_m\}$, and let $\mu' \in S_{\succ'}$ be any matching that is stable with respect to $\succ'$. Then, μ' is m-stable with respect to $\succ$.*

3.2 Structural Observations

Push up/push down operations. Note that given a profile $\succ$, the preference list of any man m can be written as $\succ_m = (\succ^L_m, \mu(m), \succ^R_m)$, where $\mu = \mathrm{DA}(\succ)$ and $\succ^L_m$ (respectively, $\succ^R_m$) is the set of women that m prefers to (respectively, finds less preferable than) $\mu(m)$. Interestingly, the DA outcome does not change even if each man m arbitrarily permutes the parts of his list above and below his DA-partner $\mu(m)$. This result, due to Huang [13], is recalled below.

PROPOSITION 3 ([13]). *Let $\succ$ be a preference profile and let $\mu := \mathrm{DA}(\succ)$. For any man $m \in M$, let $\succ'_m := (\pi^L(\succ^L_m), \mu(m), \pi^R(\succ^R_m))$, where π^L and π^R are arbitrary permutations of $\succ^L_m$ and $\succ^R_m$, respectively. Let $\succ' := \{\succ_{-m}, \succ'_m\}$, and let $\mu' := \mathrm{DA}(\succ')$. Then, $\mu' = \mu$.*

Proposition 3 considerably simplifies the structure of accomplice manipulations that we need to consider. Indeed, we can assume that any manipulated list $\succ'_m$ is such that the relative ordering of agents in the parts above and below $\mu'(m)$ is the same as under the true list $\succ_m$, where $\mu' := \mathrm{DA}(\succ')$ and $\succ' := \{\succ_{-m}, \succ'_m\}$ are the post-manipulation DA outcome and preference profile, respectively.

This observation implies that, without loss of generality, any manipulated list $\succ'_m$ can be obtained from the true list $\succ_m$ by only *push up* and *push down* operations, wherein a set of women is pushed up above the true match $\mu(m)$, and another disjoint set is pushed below $\mu(m)$. Importantly, no permutation or shuffling operation is required as part of the manipulation. Formally, starting with the true list $\succ_m = (\succ_m^L, \mu(m), \succ_m^R)$, we say that man m performs a *push up* operation for a set $X \subseteq W$ if the new list is $\succ_m^{X\uparrow} := (\succ_m^L \cup X, \mu(m), \succ_m^R \setminus X)$. Likewise, a *push down* operation of a set $Y \subseteq W$ results in $\succ_m^{Y\downarrow} := (\succ_m^L \setminus Y, \mu(m), \succ_m^R \cup Y)$.

For manipulation via push down operations only, Huang [13] has shown that the resulting matching is weakly improving for *all* men. Together, with the fact that the DA algorithm is strategyproof for the proposing side (in our case the men) [8], we get that the DA partner of the accomplice remains unchanged after a push down operation.

Proposition 4 ([8, 13]). *Let $\succ$ be the true preference profile and let $\mu := \texttt{DA}(\succ)$. For any subset of women $X \subseteq W$ and any fixed accomplice $m \in M$, let $\succ' := \{\succ_{-m}, \succ_m^{X\downarrow}\}$ and $\mu' := \texttt{DA}(\succ')$. Then, $\mu' \succeq_M \mu$ and $\mu'(m) = \mu(m)$.*

The effect of push down operations for the proposed-to side is the exact opposite, as the resulting matching makes all women weakly worse off.

Lemma 1. *Let $\succ$ be the true preference profile and let $\mu := \texttt{DA}(\succ)$. For any subset of women $X \subseteq W$, let $\succ' := \{\succ_{-m}, \succ_m^{X\downarrow}\}$ and $\mu' := \texttt{DA}(\succ')$. Then, $\mu \succeq_W \mu'$.*

Lemma 1 shows that in order to improve the partner of the woman w, the use of *push up* operations (by the accomplice) is necessary. However, it is not obvious upfront whether push up alone suffices; indeed, it is possible that the optimal strategy involves some combination of push up and push down operations. To this end, our theoretical results will show that, somewhat surprisingly, pushing up *at most one* woman achieves the desired optimal manipulation (Theorems 4.3 and 5.2). This strategy is known in the literature as *inconspicuous manipulation*, which we define next.

Inconspicuous manipulation. Given a profile $\succ$ of true preferences and any fixed accomplice m, the manipulated list $\succ'_m$ is said to be an *inconspicuous manipulation* if the list $\succ'_m$ can be derived from the true preference list $\succ_m$ by promoting exactly one woman and making no other changes. The notion of inconspicuous manipulation has been previously studied in the context of self manipulation (where w misreports herself), where it was shown that an optimal self manipulation is, without loss of generality, inconspicuous [7, 27].

4 NO-REGRET ACCOMPLICE MANIPULATION

Let us start by observing that the DA matching after an *arbitrary*, i.e., not necessarily push up, no-regret accomplice manipulation may not be stable with respect to the true preferences.

Example 4.1. Consider the following preference profile where the DA outcome is underlined.

m_1:	w_2^*	$\underline{w_1^\dagger}$	w_3	w_4	w_5
m_2:	w_1^*	$\underline{w_2^\dagger}$	w_3	w_4	w_5
$\mathbf{m_3}$:	w_1	$\underline{w_3^{*,\dagger}}$	w_4	w_2	w_5
m_4:	$\underline{w_4}$	$w_5^{*,\dagger}$	w_1	w_2	w_3
m_5:	$\underline{w_5}$	$w_4^{*,\dagger}$	w_1	w_2	w_3

w_1:	$\underline{m_1^\dagger}$	m_3	m_2^*	m_4	m_5
w_2:	$\underline{m_2^\dagger}$	m_1^*	m_3	m_4	m_5
w_3:	$\underline{m_3^{*,\dagger}}$	m_1	m_2	m_4	m_5
$\mathbf{w_4}$:	$m_5^{*,\dagger}$	m_3	m_1	m_2	$\underline{m_4}$
w_5:	$m_4^{*,\dagger}$	m_1	m_2	m_3	$\underline{m_5}$

Suppose the manipulating pair is (m_3, w_4). The DA matches after the accomplice m_3 submits the manipulated list $\succ'_{m_3} := w_4 \succ w_3 \succ w_1 \succ w_2 \succ w_5$ are marked by $*$. The manipulation results in w_4 being matched with her top choice m_5 (i.e., $\succ'_{m_3}$ is an optimal manipulation), an improvement over her true match m_4. Although m_3 does not incur regret, the manipulated matching admits a blocking pair (m_3, w_1) with respect to the true preferences. □

Notice that if instead m_3 were to submit $\succ''_{m_3} := w_4 \succ w_1 \succ w_3 \succ w_2 \succ w_5$ as his preference list in Example 4.1, then the resulting DA matching (indicated by †) would be stable with respect to the *true* preferences while still allowing w_4 to match with m_5 (i.e., $\succ''_{m_3}$ is also optimal). The manipulated list $\succ''_{m_3}$ is derived from the true list $\succ_{m_3}$ through a no-regret push up operation. Our first main result of this section (Theorem 4.2) shows that this is not a coincidence: The set of all stable matchings with respect to a profile after a no-regret *push up* operation is always contained within the stable set of the true preference profile.

Theorem 4.2 (No-regret push up is stability preserving). *Let $\succ$ be a preference profile, and let $\mu := \texttt{DA}(\succ)$. For any subset of women $X \subseteq W$ and any man m, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$, and $\mu' := \texttt{DA}(\succ')$. If m does not incur regret, then $S_{\succ'} \subseteq S_{\succ}$.*

A primary consequence of Theorem 4.2 is that the DA matching after a no-regret accomplice manipulation is weakly preferred over the true DA outcome by all women, while the opposite is true for the men.

Corollary 1. *Let $\succ$ be a preference profile and let $\mu := \texttt{DA}(\succ)$. For any man m, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$ and $\mu' := \texttt{DA}(\succ')$. If m does not incur regret, then $\mu' \succeq_W \mu$ and $\mu \succeq_M \mu'$.*

Proof. Since m does not incur regret, it follows from Theorem 4.2 that $\mu' \in S_{\succ}$. Then, from Proposition 1, we have that $\mu' \succeq_W \mu$ and $\mu \succeq_M \mu'$. □

As observed in Section 3.2, any manipulation by the accomplice can be, without loss of generality, assumed to comprise only of push up and push down operations. We will now show that combining these operations is not necessary. That is, any manipulation that is achieved by a combination of push up and push down operations can be weakly improved by a push up operation alone (Lemma 2). We note that this result does not require the no-regret assumption, and applies to the with-regret setting as well.

Lemma 2. *Let (m, w) be a manipulating pair and let $\succ$ be a preference profile. For any subsets of women $X \subseteq W$ and $Y \subseteq W$, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$ denote the preference profile after pushing up the set X, and $\succ'' := \{\succ_{-m}, \succ_m^{X\uparrow, Y\downarrow}\}$ denote the profile after pushing up X and pushing down Y in the true preference list $\succ_m$ of man m. Let $\mu := \texttt{DA}(\succ)$, $\mu' := \texttt{DA}(\succ')$, and $\mu'' := \texttt{DA}(\succ'')$. If $\mu''(w) \succ_w \mu(w)$, then $\mu'(w) \succeq_w \mu''(w)$.*

Having narrowed down the strategy space to push up operations alone, we will now turn our attention to *inconspicuous* manipulations (recall that such a manipulation involves promoting exactly one woman in the accomplice's true preference list to a higher position). We will show that any match for the manipulating woman w that can be obtained by pushing up a *set* of women can also be achieved by promoting *exactly one* woman in that set (Lemma 3). In other words, any no-regret push up operation is, without loss of generality, inconspicuous. We note that although Lemma 3 assumes no regret for the accomplice, the corresponding implication actually holds even in the with-regret setting (see Lemma 4).

Lemma 3. *Let (m, w) be a manipulating pair, and let $X \subseteq W$ be an arbitrary set of women that m can push up without incurring regret. Then, the match for w that is obtained by pushing up all women in X can also be obtained by pushing up exactly one woman in X.*

We will now use the foregoing observations to prove our main result (Theorem 4.3).

THEOREM 4.3. *If there is an optimal no-regret accomplice manipulation, then there is an optimal inconspicuous no-regret accomplice manipulation.*

PROOF. From Proposition 3 (and subsequent remarks), we know that any accomplice manipulation can be simulated via push up and push down operations. Lemma 2 shows that any beneficial manipulation that is achieved by some combination of pushing up a set $X \subseteq W$ and pushing down $Y \subseteq W$ can be weakly improved by only pushing up X. Finally, from Lemma 3, we know that any match for the manipulating woman w that is achieved by pushing up $X \subseteq W$ is also achieved by pushing up exactly one woman in X, thus establishing the desired inconspicuousness property. □

Theorem 4.3 has some interesting computational and structural implications. First, the inconspicuousness property implies a straightforward polynomial-time algorithm for computing an optimal no-regret accomplice manipulation (Corollary 2). Second, the DA matching resulting from an inconspicuous no-regret manipulation is stable with respect to the true preferences (Corollary 3). Together, these results reconcile the seemingly conflicting interests of the manipulator (who wants to compute optimal manipulation efficiently) and the central planner (who wants the resulting matching to be stable with respect to the true preferences).

Corollary 2. *An optimal no-regret accomplice manipulation strategy can be computed in $O(n^3)$ time.*

Corollary 3. *The DA outcome from an inconspicuous no-regret accomplice manipulation is stable with respect to the true preferences.*

In summary, recall from Example 4.1 that an arbitrary optimal no-regret strategy may not be stability-preserving. Nevertheless, any optimal no-regret strategy admits an equivalent inconspicuous strategy (Theorem 4.3) which indeed preserves stability (Corollary 3).

5 WITH-REGRET ACCOMPLICE MANIPULATION

No-regret manipulations come at no cost for the accomplice and thus are a viable strategic behavior (as shown in Figure 1). Yet, a more permissive strategy space may allow for the accomplice to incur some regret. Such *with-regret* manipulations may be justifiable in practice: An accomplice's idiosyncratic preference may be tolerant to a small loss in exchange of gain for the partnering woman, or a woman may persuade a man to withstand some regret by providing side-payments.

We will start by illustrating that a with-regret accomplice manipulation can be strictly more beneficial compared to its no-regret and self manipulation counterparts.

Example 5.1 (With-regret vs. no-regret). Consider the following preference profile where the DA outcome is underlined.

$\mathbf{m_1}$:	$\underline{w_4^*}$	$w_1^\dagger$	w_2	w_5	w_3	$\mathbf{w_1}$:	$m_1^\dagger$	m_2^*	$\underline{m_3}$	m_4	m_5
m_2:	$\underline{w_2}$	w_4	w_1^*	$w_5^\dagger$	w_3	w_2:	$m_3^{*,\dagger}$	m_5	m_1	$\underline{m_2}$	m_4
m_3:	$\underline{w_1}$	$w_2^{*,\dagger}$	w_4	w_3	w_5	w_3:	m_2	$\underline{m_5^*}$	m_1	$m_4^\dagger$	m_3
m_4:	w_1	$w_3^\dagger$	$\underline{w_5^*}$	w_2	w_4	w_4:	m_4	m_3	$\underline{m_1^*}$	$m_5^\dagger$	m_2
m_5:	w_1	$w_4^\dagger$	$\underline{w_3^*}$	w_5	w_2	w_5:	$\underline{m_4^*}$	$m_2^\dagger$	m_5	m_1	m_3

Suppose the manipulating pair is (m_1, w_1). The DA matching after m_1 submits the optimal no-regret[1] manipulated list $\succ'_{m_1} := w_2 \succ w_4 \succ w_1 \succ w_5 \succ w_3$ and the optimal with-regret manipulated list $\succ''_{m_1} := w_1 \succ w_4 \succ w_2 \succ w_5 \succ w_3$ are marked by $*$ and $\dagger$, respectively. Both manipulation strategies improve w_1's matching compared to truthful reporting, but w_1 strictly prefers the with-regret outcome. □

Example 5.1 highlights two key differences between optimal no-regret and with-regret manipulations. First, the matching after the *inconspicuous* with-regret manipulation (marked by $\dagger$) admits a blocking pair (m_1, w_4) with respect to the true profile $\succ$. This is in contrast to the no-regret case which is stability preserving (Theorem 4.2). Second, in contrast to Corollary 1, an optimal with-regret manipulation is not guaranteed to weakly improve or worsen the matching for *all* agents on one side; indeed the women w_3 and w_5 are strictly worse off while w_1 is strictly better off. Similarly, the man m_1 is strictly worse off while m_4 and m_5 strictly improve.

The primary distinction between no-regret and with-regret manipulation lies in the push up operations. If pushing up a set of women does not cause regret for the accomplice, then pushing up any subset thereof does not either. By contrast, if by pushing up a set of women the accomplice incurs regret, then there exists *exactly* one woman in that set who causes the same level of regret when pushed up individually. As previously mentioned, with-regret push up operations do not uniformly affect all men and all women (in contrast to Corollary 1). Moreover, the set of attained matchings after a with-regret manipulation are no longer stable with respect to true preferences (in contrast to Theorem 4.2), which makes the analysis challenging.

Despite these structural differences, we are able to prove an analogue of Lemma 3 for with-regret push up operations (Lemma 4). Our proof of this result relies on the fact that all proposals that occur when the accomplice pushes up a set of women are contained in the union of sets of proposals that occur when pushing up individual women in that set. This is relatively easy to prove for the no-regret case, since the DA matchings after these push up operations are all stable with respect to true preferences (Theorem 4.2). Although we cannot rely on the same stability result for the with-regret case, we circumvent the issue by reasoning about the sets of proposals in greater detail. The full proof of Lemma 4, along with an extensive discussion, is deferred to the appendix.

[1]To see why $\succ'_{m_1}$ is an *optimal* no-regret manipulation, note that the woman-optimal stable matching (with respect to $\succ$) matches w_1 with m_2. From Theorem 4.3 and Corollary 3, an optimal no-regret manipulation is, without loss of generality, stability preserving, and from Proposition 1, m_2 is the best stable partner for w_2.

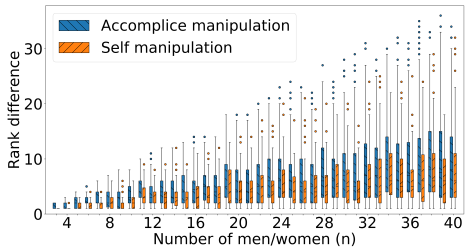

Figure 2: Comparing no-regret accomplice and self manipulation in terms of the *improvement in the rank of the matched partner* of w. The solid bars, whiskers, and dots denote the interquartile range, range excluding outliers, and outliers, respectively.

Lemma 4. *Let (m, w) be a manipulating pair, and let $X \subseteq W$ be an arbitrary set of women that m can push up (while incurring regret). Then, the match for w that is obtained by pushing up all women in X can also be obtained by pushing up exactly one woman in X.*

Subsequently, an optimal with-regret manipulation is, without loss of generality, inconspicuous. The proof is similar to that of the no-regret case (Theorem 4.3) with the only difference being the use of Lemma 4 in place of Lemma 3.

THEOREM 5.2. *If there is an optimal with-regret accomplice manipulation, then there is an optimal inconspicuous with-regret accomplice manipulation.*

Theorem 5.2 immediately implies a polynomial-time algorithm for computing an optimal with-regret accomplice manipulation. Moreover, the DA outcome from any inconspicuous with-regret accomplice manipulation is m-stable with respect to the true preferences (Proposition 2).

Corollary 4. *An optimal with-regret accomplice manipulation strategy can be computed in $O(n^3)$ time.*

6 EXPERIMENTAL RESULTS

In addition to the experiments described in Section 1, we performed a series of simulations to analyze the performance of accomplice manipulation. As for the previous experimental setup, we generated 1000 profiles uniformly at random for each value of $n \in \{3, \ldots, 40\}$ (where n is the number of men/women) and allowed any man to be chosen as an accomplice for each experiment unless stated otherwise.

Comparing the Quality of Partners. We first compare the quality of partners that a fixed strategic woman w is matched with through no-regret accomplice and self manipulation. Figure 2 illustrates the distributions of improvement (in terms of rank difference) out of only those instances where w is strictly better off through the two strategies individually. In other words, the self (respectively, accomplice) manipulation boxplots only reflect the data for when self (respectively, accomplice) manipulation is successful. It is evident that, in expectation, w is matched with better partners through no-regret accomplice manipulation.

The Fraction of Women Who Improve. We additionally compare the fraction of women who are able to improve through no-regret accomplice and self manipulation individually. Teo et al. [26] reported that 5.06% of women were able to improve using self manipulation when $n = 8$. In our experiment, this value is similarly 4.18%. However, 9.99% of women are able to improve through no-regret accomplice manipulation. As illustrated in Figure 3, the fraction of women who benefit from no-regret accomplice manipulation is consistently more than double that of self manipulation.

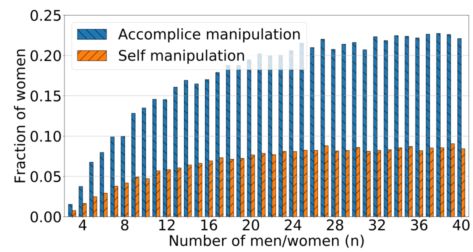

Figure 3: Comparing no-regret accomplice and self manipulation in terms of the *fraction of women who benefit*.

So far, our experiments have taken the optimistic approach of allowing the strategic woman to pick *any* man of her choice as the accomplice. However, we show that a with-regret strategy outperforms self manipulation even when a single accomplice is randomly chosen in advance, and no-regret accomplice manipulation is, on average, better than self manipulation when there is a fixed pool of four or more men to choose from. The complete discussion of this experiment, along with other experimental results, is deferred to the appendix.

7 CONCLUDING REMARKS

We showed that accomplice manipulation is a viable strategic behavior that only requires inconspicuous misreporting of preferences and is frequently more beneficial than the classical self-manipulation strategy. A natural avenue for future research is to investigate a setting with *multiple* accomplices working together to manipulate the outcome for the strategic woman. Additionally, one can think of a broader strategy space in which both the accomplice and the manipulating woman are able to misreport their preference lists simultaneously. Analyzing the benefits of such *coalitional* manipulation strategies—with or without regret—on one or both sides, and studying their structural and algorithmic properties are intriguing directions for future work.

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Appendix

8 OMITTED MATERIAL FROM SECTIONS 3.1 & 3.2

8.1 Additional Preliminaries

Lattice of stable matchings. Given any preference profile $\succ$ and any pair of stable matchings $\mu, \mu' \in S_{\succ}$, let us define their *meet* $\mu_{\wedge} := \mu \wedge \mu'$ as follows: For every man $m \in M$,

$$\mu_{\wedge}(m) = \begin{cases} \mu'(m) & \text{if } \mu(m) \succ_m \mu'(m) \\ \mu(m) & \text{otherwise,} \end{cases}$$

and for every women $w \in W$,

$$\mu_{\wedge}(w) = \begin{cases} \mu(w) & \text{if } \mu(w) \succ_w \mu'(w) \\ \mu'(w) & \text{otherwise.} \end{cases}$$

Similarly, the *join* $\mu_{\vee} := \mu \vee \mu'$ is defined as follows: For every man $m \in M$,

$$\mu_{\vee}(m) = \begin{cases} \mu(m) & \text{if } \mu(m) \succ_m \mu'(m) \\ \mu'(m) & \text{otherwise,} \end{cases}$$

and for every women $w \in W$,

$$\mu_{\vee}(w) = \begin{cases} \mu'(w) & \text{if } \mu(w) \succ_w \mu'(w) \\ \mu(w) & \text{otherwise.} \end{cases}$$

A well-known result, attributed to John Conway [15], establishes that the set of stable matchings is closed under meet and join operations.

PROPOSITION 5 ([15]). *Let $\succ$ be a preference profile and let $\mu, \mu' \in S_{\succ}$. Then, $\mu_{\wedge}, \mu_{\vee} \in S_{\succ}$.*

8.2 Self Manipulation vs. No-Regret Accomplice Manipulation

The self manipulation and no-regret accomplice manipulation strategy spaces are not contained in one another. Example 1.1 shows an instance where no-regret accomplice manipulation is better than self manipulation. In Example 8.1, we provide an instance where self manipulation is better than no-regret accomplice manipulation.

Example 8.1. Consider the following preference profile where the DA outcome is underlined.

m_1:	$\underline{w_2}$	w_3	w_1^*	w_4	w_1:	m_1^*	m_2	$\underline{m_3}$	m_4
m_2:	$\underline{w_3}$	w_2^*	w_4	w_1	w_2:	m_2^*	m_3	m_4	$\underline{m_1}$
m_3:	$\underline{w_1}$	w_3^*	w_4	w_2	w_3:	m_3^*	m_1	m_4	$\underline{m_2}$
m_4:	w_1	$\underline{w_4^*}$	w_2	w_3	w_4:	m_3	m_1	$\underline{m_4^*}$	m_2

Suppose w_1 seeks to improve her match via manipulation. The optimal self manipulation strategy for w_1 is $\succ'_{w_1} = m_4 \succ m_3 \succ m_1 \succ m_2$, which allows her to match with her top choice m_1 (the new DA matching is marked by $*$). Regardless of the choice of accomplice, the optimal no-regret accomplice manipulation strategy, on the other hand, is truth-telling. □

8.3 Proof of Proposition 2

PROPOSITION 2. *Let $\succ$ denote the true preference profile. For any man m, let $\succ' := \{\succ_{-m}, \succ'_m\}$, and let $\mu' \in S_{\succ'}$ be any matching that is stable with respect to $\succ'$. Then, μ' is m-stable with respect to $\succ$.*

PROOF. Suppose, for contradiction, that μ' is not m-stable with respect to $\succ$. Then, there must exist a man-woman pair (m', w') that blocks μ' with respect to $\succ$ such that $m' \neq m$, i.e., $w' \succ_{m'} \mu'(m')$ and $m' \succ_{w'} \mu'(w')$. Since m is the only agent whose preferences differ between $\succ$ and $\succ'$, we have that $\succ_{m'} = \succ'_{m'}$ and $\succ_{w'} = \succ'_{w'}$. Thus, $w' \succ'_{m'} \mu'(m')$ and $m' \succ'_{w'} \mu'(w')$, implying that the pair (m', w') blocks μ' with respect to $\succ'$, which contradicts the assumption that $\mu' \in S_{\succ'}$. Thus, μ' must be m-stable with respect to the true profile $\succ$. □

8.4 Proof of Lemma 1

Lemma 1. *Let $\succ$ be the true preference profile and let $\mu := \texttt{DA}(\succ)$. For any subset of women $X \subseteq W$, let $\succ' := \{\succ_{-m}, \succ_m^{X\downarrow}\}$ and $\mu' := \texttt{DA}(\succ')$. Then, $\mu \succeq_W \mu'$.*

PROOF. Suppose, for contradiction, that there exists a woman w' such that $m' \succ_{w'} \mu(w')$, where $m' := \mu'(w')$. We infer that $w' \not\succ_{m'} \mu(m')$, otherwise the stability of μ with respect to $\succ$ is compromised. Since $m' \neq \mu(w')$, we have that $w' \neq \mu(m')$, and therefore $\mu(m') \succ_{m'} w'$. However, Proposition 4 guarantees $\mu' \succeq_M \mu$, thus posing a contradiction. □

9 OMITTED MATERIAL FROM SECTION 4

9.1 Proof of Theorem 4.2

THEOREM 4.2 (NO-REGRET PUSH UP IS STABILITY PRESERVING). *Let $\succ$ be a preference profile, and let $\mu := \texttt{DA}(\succ)$. For any subset of women $X \subseteq W$ and any man m, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$, and $\mu' := \texttt{DA}(\succ')$. If m does not incur regret, then $S_{\succ'} \subseteq S_{\succ}$.*

PROOF. Suppose, for contradiction, that there exists a matching $\phi \in S_{\succ'} \setminus S_{\succ}$. Then, there must be a pair (m', w') that blocks ϕ with respect to $\succ$. It follows from Proposition 2 that $m' = m$. Thus, $w' \succ_m \phi(m)$ and $m \succ_{w'} \phi(w')$.

From Proposition 1, we have that $\mu'(m) \succeq'_m \phi(m)$. Since m does not incur regret, we have $\mu(m) = \mu'(m)$, and thus, $\mu(m) \succeq'_m \phi(m)$. All women below $\mu(m)$ in $\succ'_m$ are also below $\mu(m)$ in $\succ_m$ by the push up assumption. Since $\mu(m) \succeq'_m \phi(m)$, this implies that all women below $\phi(m)$ in $\succ'_m$ are also below $\phi(m)$ in $\succ_m$. Thus, if $\phi(m) \succ'_m w'$, then $\phi(m) \succ_m w'$, which contradicts the blocking pair condition above. Therefore, we must have that $w' \succ'_m \phi(m)$ (note that $w' \neq \phi(m)$ by the blocking pair condition).

Furthermore, since $\succ'_{w'} = \succ_{w'}$, the blocking pair condition also implies that $m \succ'_{w'} \phi(w')$. Combined with the condition $w' \succ'_m \phi(m)$, this contradicts the assumption that $\phi \in S_{\succ'}$. Thus, $S_{\succ'} \subseteq S_{\succ}$. □

9.2 Proof of Lemma 2

Lemma 2. *Let (m, w) be a manipulating pair and let $\succ$ be a preference profile. For any subsets of women $X \subseteq W$ and $Y \subseteq W$, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$ denote the preference profile after pushing up the set X, and $\succ'' := \{\succ_{-m}, \succ_m^{X\uparrow, Y\downarrow}\}$ denote the profile after pushing up*

X and pushing down Y in the true preference list $\succ_m$ of man m. Let $\mu := \texttt{DA}(\succ)$, $\mu' := \texttt{DA}(\succ')$, and $\mu'' := \texttt{DA}(\succ'')$. If $\mu''(w) \succ_w \mu(w)$, then $\mu'(w) \succeq_w \mu''(w)$.

Proof. Our proof will use case analysis based on whether or not $\mu'(m) = \mu(m)$.

Case I (when $\mu'(m) = \mu(m)$): The list $\succ''_m$ can be considered as being derived from $\succ'_m$ via a push down operation on the set Y (see Figure 4). From Lemma 1, we know that a push down operation is weakly worse for all women; thus, in particular, we get $\mu'(w) \succeq_w \mu''(w)$, as desired. Note that the relative ordering of X and Y above $\mu'(m)$ in the list $\succ'_m$ is not important in light of Proposition 3.

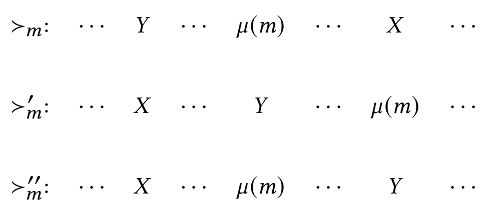

Figure 4: An illustration of man m's preference lists under the profiles $\succ$, $\succ'$, and $\succ''$ in the proof of Lemma 2.

Case II (when $\mu'(m) \neq \mu(m)$): Suppose $\mu'(m) \in X$. Then, the list $\succ''_m$ can be considered as being derived from $\succ'_m$ via a permutation of the women below $\mu'(m)$ (see Figure 4). By Proposition 3, this implies that $\mu' = \mu''$, and in particular $\mu'(w) = \mu''(w)$, as desired.

Therefore, for the remainder of the proof, let us assume that $\mu'(m) \notin X$. Since $\succ'_m$ is derived from $\succ_m$ via a push up operation on the set X, and since $\mu'(m) \neq \mu(m)$ by assumption, we have that $\mu(m) \succ_m \mu'(m)$. By Proposition 3, we can assume, without loss of generality, that $\mu'(m)$ is positioned immediately below $\mu(m)$ in the list $\succ_m$. By construction, the same property also holds for the lists $\succ'_m$ and $\succ''_m$. Thus, $\succ''_m$ can be considered as being obtained from $\succ'_m$ via a push down operation on the set Y (note that this operation is defined with respect to $\mu'(m)$). By Lemma 1, we have $\mu'(w) \succeq_w \mu''(w)$, as desired. □

9.3 Proof of Lemma 3

Lemma 3. *Let (m, w) be a manipulating pair, and let $X \subseteq W$ be an arbitrary set of women that m can push up without incurring regret. Then, the match for w that is obtained by pushing up all women in X can also be obtained by pushing up exactly one woman in X.*

Preliminaries for the proof of Lemma 3: Let $\succ$ be a true preference profile. Given an accomplice m and a set of women $X = \{w_a, w_b, w_c, \ldots\}$ such that $\mu(m) \succ_m w_x$ for all $w_x \in X$, we define $\succ^a_m$ as the list derived from $\succ_m$ where m pushes up w_a, $\succ^b_m$ as the list derived from $\succ_m$ where m pushes up w_b, etc., and $\succ^X_m$ as the list where m pushes up all women in X; the corresponding profiles are $\succ^a, \succ^b, \succ^c, \ldots, \succ^X$. Additionally, let $\mu^a := \texttt{DA}(\succ^a), \mu^b := \texttt{DA}(\succ^b), \mu^c := \texttt{DA}(\succ^c), \ldots, \mu^X := \texttt{DA}(\succ^X)$. Note that the placement of women being pushed up above $\mu(m)$ in $\succ^a_m, \succ^b_m, \succ^c_m, \ldots, \succ^X_m$ does not affect $\mu^a, \mu^b, \mu^c, \ldots, \mu^X$ by Proposition 3.

Example 9.1. Suppose $X = \{w_a, w_b\}$. Figure 5 illustrates the possible configurations of m's preference lists under the profiles $\succ$, $\succ^a$, $\succ^b$, and $\succ^X$.

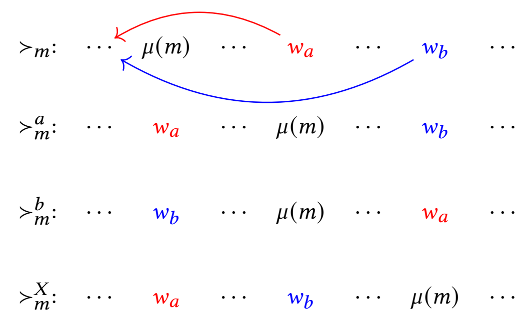

Figure 5: An illustration of man m's preference lists under the profiles $\succ$, $\succ^a$, $\succ^b$, and $\succ^X$ when $X = \{w_a, w_b\}$.

It can be shown that m does not incur regret under any of the profiles $\succ^a, \succ^b$, and so on (Lemma 5).

Lemma 5. *Let $X = \{w_a, w_b, w_c, \ldots\}$ be an arbitrary finite set of women that the accomplice m can push up without incurring regret. Then, for every $w_x \in X$, m does not incur regret under the matching $\mu^x := \texttt{DA}(\succ^x)$, where $\succ^x := \{\succ_{-m}, \succ^{w_x\uparrow}_m\}$.*

Proof. We will prove the lemma for the fixed profile $\succ^a$ (an identical argument works for other profiles).

Suppose, for contradiction, that m incurs regret in the profile $\succ^a$. That is, $\mu(m) \succ_m \mu^a(m)$ where $\mu^a := \texttt{DA}(\succ^a)$. From Lemma 7 (see Section 10.1), we get that $\mu^a(m) = w_a$. Further, using Proposition 3, we can assume, without loss of generality, that w_a is positioned immediately below $\mu(m)$ in the true list $\succ_m$, and immediately above it in the manipulated lists $\succ^a_m$, as well as $\succ^X_m$. This implies that the transition from $\succ^a_m$ to $\succ^X_m$ is a *with-regret* push up operation involving the promotion of $X \setminus \{w_a\}$ (since, according to the list $\succ^a_m$, the new partner $\mu(m)$ is strictly worse than w_a). Again, from Lemma 7, it follows that $\mu^X(m) \in X \setminus \{w_a\}$. By the no-regret assumption for the set X, we know that $\mu^X(m) = \mu(m)$. This, however, is a contradiction since all women in $X \setminus \{w_a\}$ are placed below $\mu(m)$ in the list $\succ_m$, and hence must be different from it. □

Given any profile $\succ$, let $P_\succ$ denote the set of all proposals that occur in the execution of the $\texttt{DA}$ algorithm on $\succ$. Formally, for any man $m_i \in M$ and woman $w_j \in W$, the ordered pair (m_i, w_j) belongs to the set $P_\succ$ if m_i proposes to w_j during the execution of $\texttt{DA}$ algorithm on the profile $\succ$.

Lemma 6. *Let $X = \{w_a, w_b, w_c, \ldots\}$ be an arbitrary finite set of women that the accomplice m can push up without incurring regret. Then, any proposal that occurs under $\succ^X$ also occurs under at least one of the profiles $\succ^a, \succ^b, \succ^c, \ldots$.*

Proof. Suppose, for contradiction, that (m_1, w_1) is the *first* proposal to occur during the $\texttt{DA}$ execution on $\succ^X$ such that it is not an element of $P_{\succ^a} \cup P_{\succ^b} \cup P_{\succ^c} \cup \ldots$. Note that the proposals made by the accomplice m in $P_{\succ^X}$ are only to the women above and including $\mu(m)$ in $\succ_m$ as well as the women in $\{w_a, w_b, w_c, \ldots\}$, all of whom he proposes to in $P_{\succ^a} \cup P_{\succ^b} \cup P_{\succ^c} \cup \ldots$. Thus, $m_1 \neq m$, implying that m_1 is a truthful agent.

Since men propose in decreasing order of their preference, m_1 must have been rejected by a woman, say w_2, whom he ranks immediately above w_1 in $\succ_{m_1}$, before proposing to w_1 under $\succ^X$. Further, m_1 must propose to w_2 under at least one of the profiles $\succ^a, \succ^b, \succ^c, \dots$, and must not be rejected by her (otherwise, he will propose to w_1). Thus, m_1 must be matched with w_2 under at least one of the matchings $\mu^a, \mu^b, \mu^c, \dots$. Without loss of generality, let us assume that $\mu^a(m_1) = w_2$.

Since w_2 is not matched with m_1 under μ^X, she must have received a more preferable proposal from some man, say m_2; thus, $m_2 \succ_{w_2} m_1$. Due to our assumption that (m_1, w_1) is the first proposal during the DA execution on $\succ^X$ that does not occur in $P_{\succ^a} \cup P_{\succ^b} \cup P_{\succ^c} \cup \dots$, m_2 must have proposed to w_2 under at least one of the profiles $\succ^a, \succ^b, \succ^c, \dots$. Without loss of generality, he proposes under $\succ^b$. Since women match with their best proposers, $\mu^b(w_2) \succeq_{w_2} m_2$. This, combined with $m_2 \succ_{w_2} m_1 = \mu^a(w_2)$, we get $\mu^b(w_2) \succ_{w_2} \mu^a(w_2)$.

We infer that m_1 does not propose to w_2 under $\succ^b$, since otherwise w_2 (eventually) rejects m_1 causing him to propose to w_1, which would contradict our assumption that $(m_1, w_1) \notin P_{\succ^a} \cup P_{\succ^b} \cup P_{\succ^c} \cup \dots$. Thus, $\mu^b(m_1) \succ_{m_1} w_2 = \mu^a(m_1)$.

Notice that the profiles $\succ^a$ and $\succ^b$ are obtained from the true preference profile $\succ$ by no-regret push up operations. Therefore, from Lemma 5, we get that the matchings μ^a and μ^b are stable with respect to the true preference profile (i.e., $\mu^a, \mu^b \in S_\succ$).

Now consider the *join* $\mu_\vee := \mu^a \vee \mu^b$ of the two matchings with respect to the true preference profile $\succ$, wherein each man is associated with his more preferred partner between μ^a and μ^b, and each woman is associated with her less preferred partner (refer to Section 8.1 for the relevant definitions). Thus, m_1 is associated with $\mu^b(m_1) \neq w_2$ and w_2 is associated with $\mu^a(w_1) = m_1$. The resulting assignment $\mu_\vee$ is not a valid matching, which contradicts the stable lattice result (Proposition 5). □

We are now ready to prove Lemma 3.

Lemma 3. *Let (m, w) be a manipulating pair, and let $X \subseteq W$ be an arbitrary set of women that m can push up without incurring regret. Then, the match for w that is obtained by pushing up all women in X can also be obtained by pushing up exactly one woman in X.*

Proof. Let $m_X := \mu^X(w)$, where $\mu^X := \text{DA}(\succ^X)$. From Lemma 5, we know that the accomplice m does not incur regret under any of the matchings $\mu^a, \mu^b, \mu^c, \dots$, and therefore $\mu^a(m) = \mu^b(m) = \mu^c(m) = \dots = \mu(m)$. If $m_X = m$, then the lemma follows trivially since m is matched with his μ^X-partner, namely w, under each of the matchings $\mu^a, \mu^b, \mu^c, \dots$, and therefore the μ^X-partner of w can also be achieved under any of the profiles $\succ^a, \succ^b, \succ^c, \dots$. Thus, for the remainder of the proof, we will assume that $m_X \neq m$; in other words, m_X is a truthful agent.

Suppose, for contradiction, that m_X is not matched to w under any of the profiles $\succ^a, \succ^b, \succ^c, \dots$. Starting from the profile $\succ^a$, we can obtain the profile $\succ^X$ via a no-regret push up operation of the set $X \setminus \{w_a\}$ in the list $\succ^a_m$ of the accomplice. Therefore, from Corollary 1, we have that $\mu^a(m_X) \succeq_{m_X} \mu^X(m_X) = w$, where $\mu^a := \text{DA}(\succ^a)$. Since $\mu^a(m_X) \neq w$ by the contradiction assumption, we further obtain that $\mu^a(m_X) \succ_{m_X} w$. By a similar argument, we get that the women $\mu^b(m_X), \mu^c(m_X), \dots$ are also placed above w in the list $\succ_{m_X}$.

Since m_X is matched with w under μ^X, he must propose to w during the execution of DA algorithm on $\succ^X$, i.e., $(m_X, w) \in P_{\succ^X}$. From Lemma 6, we have that $(m_X, w) \in P_{\succ^a} \cup P_{\succ^b} \cup P_{\succ^c} \dots$. Since m_X is a truthful agent, his preference list remains unchanged, and therefore he ranks the women $\mu^a(m), \mu^b(m_X), \mu^c(m_X), \dots$ strictly above w under each of the profiles $\succ^a, \succ^b, \succ^c, \dots$. This, however, poses a contradiction since men propose in decreasing order of their preference. □

9.4 Proof of Corollary 2

Corollary 2. *An optimal no-regret accomplice manipulation strategy can be computed in $O(n^3)$ time.*

Proof. (sketch) The algorithm simply promotes each woman that is below $\mu(m)$ in the accomplice's true preference list to some position above $\mu(m)$ and checks the DA outcome. The total number of such checks is $O(n)$, and for each check, running the DA algorithm takes $O(n^2)$ time. □

9.5 Proof of Corollary 3

Corollary 3. *The DA outcome from an inconspicuous no-regret accomplice manipulation is stable with respect to the true preferences.*

Proof. (sketch) An inconspicuous manipulation is a special case of a push up operation, which was shown in Theorem 4.2 to be stability preserving. □

10 OMITTED MATERIAL FROM SECTION 5

10.1 Proof of Theorem 5.2

Theorem 5.2. *If there is an optimal with-regret accomplice manipulation, then there is an optimal inconspicuous with-regret accomplice manipulation.*

Recall from Lemma 3 that the match for the manipulating woman w obtained by a no-regret push up operation of a set $X \subseteq W$ by the accomplice can also be achieved by pushing up exactly one woman in X. The following result (Lemma 4) establishes the with-regret analogue of this result.

Lemma 4. *Let (m, w) be a manipulating pair, and let $X \subseteq W$ be an arbitrary set of women that m can push up (while incurring regret). Then, the match for w that is obtained by pushing up all women in X can also be obtained by pushing up exactly one woman in X.*

Preliminaries for the proof of Lemma 4: The relevant notation is similar to that used in the proof of Lemma 3, which we recall below for the sake of completeness.

Let $\succ$ be a true preference profile. Given an accomplice m and a set of women $X = \{w_a, w_b, w_c, \dots\}$ such that $\mu(m) \succ_m w_x$ for all $w_x \in X$, we define $\succ^a_m$ as the list derived from $\succ_m$ where m pushes up w_a, $\succ^b_m$ as the list derived from $\succ_m$ where m pushes up w_b, etc., and $\succ^X_m$ as the list where m pushes up all women in X; the corresponding profiles are $\succ^a, \succ^b, \succ^c, \dots, \succ^X$. Additionally, let $\mu^a := \text{DA}(\succ^a)$, $\mu^b := \text{DA}(\succ^b)$, $\mu^c := \text{DA}(\succ^c), \dots, \mu^X := \text{DA}(\succ^X)$.

We will now show that under a with-regret push up operation, the DA algorithm matches the accomplice to one of the women he pushes up (Lemma 7). In particular, if the accomplice promotes only one woman and incurs regret, then he must be matched to her in the resulting matching.

Lemma 7. *Let $\succ$ be a true preference profile and $\mu := \texttt{DA}(\succ)$. For any fixed man m and any subset $X \subseteq W$ of women, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$ denote the preference profile after pushing up the women in X in $\succ_m$ and let $\mu' := \texttt{DA}(\succ')$. If m incurs regret (i.e., if $\mu(m) \succ_m \mu'(m)$), then $\mu'(m) \in X$.*

Proof. Suppose, for contradiction, that $\mu'(m) \notin X$. Because of Proposition 3, without loss of generality, we have that the set X is placed immediately below $\mu(m)$ in the true list $\succ_m$. By strategyproofness of the DA algorithm, m cannot be matched under μ' to any woman who, according to his true list, is preferred over $\mu(m)$, i.e., $\mu'(m) \notin \{z \in W : z \succ_m \mu(m)\}$. By the contradiction assumption, m cannot be matched with any woman in X, and because of the with-regret assumption, m cannot be matched with $\mu(m)$ either. Therefore, the woman $y := \mu'(m)$ must be such that $\mu(m) \succ'_m y$, which, by the push up assumption, implies that $\mu(m) \succ_m y$.

Once again, due to Proposition 3. we can assume, without loss of generality, that y is placed immediately below the set X in the true list $\succ_m$. This, in turn, implies that y is immediately below $\mu(m)$ in the manipulated list $\succ_m^{X\uparrow}$. Therefore, starting with the list $\succ_m^{X\uparrow}$, one can obtain the true list $\succ_m$ simply by permuting the agents that are above y. Proposition 3 would then imply that the partner of m does not change in the process, i.e., $\mu'(m) = \mu(m)$, which is a contradiction. Hence, we must have $\mu'(m) \in X$. □

Lemma 7 implies that the accomplice m matches with some woman in X under the matching μ^X; say $\mu^X(m) = w_a$. We will now show that m is matched with w_a under the matching μ^a as well (Lemma 8). Notice that in light of Proposition 3, we can assume, without loss of generality, that in the lists $\succ_m^X$ and $\succ_m^a$, the woman w_a is placed immediately above $\mu(m)$. That is, w_a is the least-preferred woman in X according to the list $\succ_m^X$.

Lemma 8. *Let $w_a \in X$ be the woman who the accomplice m matches with under μ^X. Then, m also matches with w_a under μ^a, where $\mu^a := \texttt{DA}(\succ^a)$ and $\succ^a := \{\succ_{-m}, \succ_m^{w_a\uparrow}\}$.*

Proof. Starting with the profile $\succ^X$, we can obtain $\succ^a$ by pushing down all women in $X \setminus \{w_a\}$ (recall from the aforementioned observation that w_a is the least-preferred woman in X according to the list $\succ_m^X$). Then, from Proposition 4, we get that $\mu^a(m) = \mu^X(m)$. □

From Lemma 8, it follows that $\succ^a$ is a with-regret profile. By contrast, Lemma 9 will show that the rest of the profiles $\succ^b, \succ^c \dots$ do not cause regret for the accomplice.

Lemma 9. *Let $w_a \in X$ be the woman who the accomplice m matches with under μ^X. Then, for any $w_z \in X \setminus \{w_a\}$, m does not incur regret under the profile $\succ^z := \{\succ_{-m}, \succ_m^{w_z\uparrow}\}$.*

Proof. Let $\tilde{X} := X \setminus \{w_a\}$, and let $\succ^{\tilde{X}} := \{\succ_{-m}, \succ_m^{\tilde{X}\uparrow}\}$ be the profile where m pushes up all women in $\tilde{X}$ starting from the true list $\succ_m$.

We claim that m must match with the woman $\mu(m)$ under the matching $\mu^{\tilde{X}} := \texttt{DA}(\succ^{\tilde{X}})$. Indeed, if that is not the case, then promoting $\tilde{X}$ is a with-regret push up operation. Then, from Lemma 7, the man m must be matched with some woman in $\tilde{X}$ under $\mu^{\tilde{X}}$. This, however, would contradict the strategyproofness of DA algorithm, as m is able to strictly improve in going from the "true" list $\succ_m^X$ (where he is matched with w_a, who is the least-preferred woman in X according to the list $\succ_m^X$) to the "manipulated" list $\succ^{\tilde{X}}$ (where his partner is some woman in $\tilde{X}$).

Thus, m must be matched with the woman $\mu(m)$ under $\mu^{\tilde{X}}$, implying that promoting $\tilde{X}$ is a no-regret push up operation. Lemma 5 now implies that for every $w_z \in \tilde{X}$, $\succ^z := \{\succ_{-m}, \succ_m^{w_z\uparrow}\}$ is also a no-regret profile, as desired. □

It can also be shown that if the accomplice m pushes up all women in $X \setminus \{w_a\}$ simulataneously, then he does not incur regret (Lemma 10).

Lemma 10. *Let $\succ^{\bar{X}}$ be the profile obtained by pushing up $\bar{X} := X \setminus \{w_a\}$ in the accomplice m's true preference list. Then, m does not incur regret under $\succ^{\bar{X}}$ (i.e, m matches with $\mu(m)$ under $\succ^{\bar{X}}$).*

Proof. Suppose, for contradiction, that $\succ^{\bar{X}}$ is a with-regret profile. Then, from Lemmas 7 and 8, we have that for some $w_z \in \bar{X}$, the profile $\succ^z := \{\succ_{-m}, \succ_m^{w_z\uparrow}\}$ is also with-regret. This, however, contradicts the implication of Lemma 9 that $\succ^z$ is no-regret for every $w_z \in \bar{X}$. □

Recall that given any profile $\succ$, $P_\succ$ denotes the set of all proposals that occur in the execution of the DA algorithm on $\succ$. Formally, for any man $m_i \in M$ and woman $w_j \in W$, the ordered pair (m_i, w_j) belongs to the set $P_\succ$ if m_i proposes to w_j during the execution of DA algorithm on the profile $\succ$. Our next result (Lemma 11) shows that the set of proposals that occur under a true preference profile $\succ$ is contained in the set of proposals that occur under a profile $\succ'$ that is obtained via a no-regret push up operation on $\succ$.

Lemma 11. *Let $\succ$ be a preference profile and $\mu := \texttt{DA}(\succ)$. For any fixed man m, let $\succ' = \{\succ_{-m}, \succ_m^{X\uparrow}\}$ and $\mu' = \texttt{DA}(\succ')$ such that m does not incur regret (i.e., $\mu'(m) = \mu(m)$). Then, $P_\succ \subseteq P_{\succ'}$.*

Proof. Since m does not incur regret under $\succ'$, we have that $\mu \succeq_M \mu'$ (Corollary 1). Under the DA algorithm, men propose in decreasing order of their preference. Therefore, any proposal made by a truthful man under $\succ$ is also made under $\succ'$. Furthermore, the push up assumption implies that the accomplice m proposes to the women in $\succ_m^L$ (i.e., the woman strictly preferred by m over $\mu(m)$ according to his true list $\succ_m$) under $\succ^X$ as well. □

Let $P_{\succ\setminus\succ'} := P_\succ \setminus P_{\succ'}$ denote the set of proposals that occur under the profile $\succ$ but not under $\succ'$. Our next result (Lemma 12) shows that the set $P_{\succ^z\setminus\succ^X}$, where $\succ^z$ is a no-regret profile obtained by pushing up some woman $w_z \in X \setminus \{w_a\}$ (as established in Lemma 9), is contained in the set $P_\succ$.

Lemma 12. *For any woman $w_z \in X \setminus \{w_a\}$, let $\succ^z$ be the no-regret profile obtained by pushing up w_z (as discussed in Lemma 9). Then, $P_{\succ^z\setminus\succ^X} \subseteq P_\succ$.*

Proof. We start by showing that any proposal in $P_{\succ^z\setminus\succ^X}$ must occur after m proposes to $\mu(m)$ during the DA execution on $\succ^z$. Suppose, for contradiction, that this is not true. Then, let (m_1, w_1) be the *first* proposal during the DA execution on $\succ^z$ such that it does not belong to $P_{\succ^X}$ and occurs before $(m, \mu(m))$. Note that the proposals made by the accomplice m before $\mu(m)$ under $\succ^z$ are only to the women above and including $\mu(m)$ in $\succ_m$ as well as w_z, all of whom he also proposes to under $\succ^z$. Thus, $m_1 \neq m$, implying that m_1 is a truthful agent.

Since men propose in decreasing order of their preference and it is assumed that $(m_1, w_1) \notin P_{\succ^X}$, m_1 must have been rejected by $w_2 := \mu^X(m_1)$ under $\succ^z$ before proposing to w_1. Then, under $\succ^z$, w_2 must have received a proposal from some man, say m_2, such that $m_2 \succ_{w_2} m_1$. We assumed (m_1, w_1) to be the first proposal during the DA execution on $\succ^z$ to not belong to $P_{\succ^X}$. Since (m_2, w_2) occurs before (m_1, w_1) under $\succ^z$, we must have that (m_2, w_2) also occurs under $\succ^X$. We already established that $m_2 \succ_{w_2} m_1$ and know that women are truthful. Since women match with their favorite proposers, this implies that w_2 does not match with m_1 under $\succ^X$. However, this contradicts the statement that $w_2 = \mu^X(m_1)$.

We proceed by showing that any proposal that occurs after m proposes to $\mu(m)$ during the DA execution on $\succ^z$ must also occur in $P_\succ$. Suppose, for contradiction, that this is not true. Then, let (m_1, w_1) be the *last* proposal during the DA execution on $\succ^z$ such that it does not belong to $P_\succ$ and occurs after $(m, \mu(m))$. We infer that w_1 accepts and matches with m_1 under $\succ^z$, since otherwise m_1 must make additional proposals, contradicting our assumption that (m_1, w_1) is the last proposal to not occur in $P_\succ$. Note that m does not propose to anyone after $\mu(m)$ by the no-regret assumption of $\succ^z$. Thus, $m_1 \neq m$, implying that m_1 is a truthful agent.

Let m_2 be the man w_1 is temporarily engaged to before she receives a proposal from m_1 under $\succ^X$. We assumed that $w_2 = \mu^X(m_1)$ which means that w_1 rejects m_2 in favor of m_1 under $\succ^z$, causing m_2 to propose to the next woman in $\succ^z_{m_2}$, say w_2. Since $P_\succ \subseteq P_{\succ^z}$ (Lemma 11), we know that $(\mu(w_1), w_1) \in P_{\succ^z}$. We also defined m_2 to be w_1's favorite proposer under $\succ^z$ before matching with m_1, which implies that $m_2 \succeq_{w_1} \mu(w_1)$. If $w_1 \succ_{m_2} \mu(m_2)$, then the pair (m_2, w_1) blocks μ with respect to $\succ$. Thus, we have that $\mu(m_2) \succeq_{m_2} w_1$. Note that the accomplice m does not make any proposals after (m_1, w_1) under $\succ^z$, since (m_1, w_1) occurs after $(m, \mu(m))$ and $\succ^z$ is a no-regret profile. We know that m_2 proposes to w_2 after (m_1, w_1), implying that $m_2 \neq m$. Therefore, $\mu(m_2) \succeq_{m_2} w_1 \succ_{m_2} w_2$, and m_2 does not propose to w_2 under $\succ$. However, this contradicts the assumption that (m_1, w_1) is the last proposal during the DA execution on $\succ^X$ to not belong to $P_\succ$.

Thus, we have shown that (1) any proposal that belongs to $P_{\succ^z \setminus \succ^X}$ occurs after $(m, \mu(m))$ during the DA execution on $\succ^z$, and (2) any proposal that occurs after $(m, \mu(m))$ during the DA execution on $\succ^z$ belongs to $P_\succ$. These two statements imply that $P_{\succ^z \setminus \succ^X} \subseteq P_\succ$. □

Our next result (Lemma 13) shows that the set $P_{\succ^X \setminus \succ^{\bar{X}}}$ is contained in the set $P_{\succ^a}$.

Lemma 13. *Let $\succ^{\bar{X}}$ be the profile obtained by pushing up $\bar{X} := X \setminus \{w_a\}$ in the accomplice m's true preference list. Then, $P_{\succ^X \setminus \succ^{\bar{X}}} \subseteq P_{\succ^a}$.*

PROOF. For this proof, let $\succ^a$ be the "true" preference profile. Remember, from Lemma 10, that m matches with $\mu(m)$ under $\succ^{\bar{X}}$. Thus, $\succ^{\bar{X}}$ is a with-regret profile with respect to $\succ^a$, and is obtained by pushing up $\hat{X} := \bar{X} \cup \{\mu(m)\}$ in $\succ^a_m$. On the other hand, m matches with w_a under $\succ^a$ and $\succ^X$ (Lemma 8). Thus, $\succ^X$ is a no-regret profile with respect to $\succ^a$, and is obtained by pushing up $\bar{X} = \hat{X} \setminus \{\mu(m)\} = X \setminus \{w_a\}$.

For any woman $w_z \in \bar{X}$, let $\succ^{az}$ be the profile obtained by pushing up w_z in $\succ^a_m$. It is easy to see that $\succ^{az}$ is a no-regret profile with respect to $\succ^a$. Indeed, suppose $\succ^{az}$ is with-regret. Then, m must match with w_z under $\succ^{az}$ (Lemma 7). In light of Proposition 3 we can assume, without loss of generality, that m ranks the set $\bar{X}$ strictly above w_a in the list $\succ^X_m$. Given profile $\succ^X$, m could then manipulate by submitting $\succ^{az}_m$ in order to match with w_z. However, this would contradict strategyproofness of the DA algorithm since $w_z \succ_m w_a$. Therefore, $\succ^{az}$ is no-regret with respect to $\succ^a$.

Given this observation, from Lemma 12, we get that for any woman $w_z \in \bar{X} = \hat{X} \setminus \{\mu(m)\}$, $P_{\succ^{az}} \setminus P_{\succ^{\bar{X}}} \subseteq P_{\succ^a}$. Since m matches with w_a under $\succ^{az}$ and $\succ^X$, we have that $\succ^{az}$ is no-regret with respect to $\succ^X$. Thus, we invoke Lemma 6 to claim that any proposal that occurs under $\succ^X$ is contained in $P_{\succ^{az_1}} \cup P_{\succ^{az_2}} \cup \cdots \cup P_{\succ^{az_k}}$, where $\bar{X} := \{w_{z_1}, w_{z_2}, \ldots, w_{z_k}\}$. Since $\{P_{\succ^{az_1}} \cup P_{\succ^{az_2}} \cup \cdots \cup P_{\succ^{az_k}}\} \setminus P_{\succ^{\bar{X}}} \subseteq P_{\succ^a}$, we get that $P_{\succ^X \setminus \succ^{\bar{X}}} \subseteq P_{\succ^a}$. □

Recall from Lemma 6 that, in the no-regret setting, any proposal that occurs during the DA execution on the profile $\succ^X$ must also occur during the DA execution on at least one of the profiles $\succ^a, \succ^b, \succ^c, \ldots$. Our next result (Lemma 14) establishes the with-regret analogue of this result.

Lemma 14. *Let $X = \{w_a, w_b, w_c, \ldots\}$ be an arbitrary finite set of women such that the accomplice m incurs regret after pushing up X. Then, any proposal that occurs under $\succ^X$ also occurs under at least one of the profiles $\succ^a, \succ^b, \succ^c, \ldots$.*

PROOF. Let $\succ^{\bar{X}}$ be the profile after m pushes up $\bar{X} := X \setminus \{w_a\}$ in his true preference list. Then, from Lemma 13, we know that $P_{\succ^X} \setminus P_{\succ^{\bar{X}}} \subseteq P_{\succ^a}$. Additionally, from Lemma 10, we know that $\succ^{\bar{X}}$ is a no-regret profile. It then follows from Lemma 6 that $P_{\succ^{\bar{X}}} \subseteq P_{\succ^b} \cup P_{\succ^c} \cup \ldots$.

Any proposal that occurs under $\succ^X$ is contained in either $P_{\succ^X} \setminus P_{\succ^{\bar{X}}}$ or $P_{\succ^{\bar{X}}}$. By combining the aforementioned observations, we get that any such proposal is contained in $P_{\succ^a} \cup P_{\succ^b} \cup P_{\succ^c} \cup \ldots$, as desired. □

We are now ready to prove Lemma 4.

Lemma 4. *Let (m, w) be a manipulating pair, and let $X \subseteq W$ be an arbitrary set of women that m can push up (while incurring regret). Then, the match for w that is obtained by pushing up all women in X can also be obtained by pushing up exactly one woman in X.*

PROOF. Let $m_X := \mu^X(w)$, where $\mu^X := \text{DA}(\succ^X)$. We assume that the accomplice m matches with a woman $w_a \in X$ (Lemma 7) under $\succ^X$. From Lemma 8, we know that he also matches with w_a under $\succ^a$. If $m_X = m$, then the lemma follows trivially since m is matched with his μ^X-partner, namely w, under the matching μ^a, and therefore the μ^X-partner of w can also be achieved through profile $\succ^a$. Thus, for the remainder of the proof, we assume that $m_X \neq m$; in other words, m_X is a truthful agent.

Suppose, for contradiction, that m_X is not matched to w under any of the profiles $\succ^a, \succ^b, \succ^c, \ldots$. Starting from the with-regret profile $\succ^a$ (as established in Lemma 8), we can obtain the profile $\succ^X$ via a no-regret push up operation of the set $X \setminus \{w_a\}$ in the list $\succ_m^a$ of the accomplice. Therefore, from Corollary 1, we have that $\mu^a(m_X) \succeq_{m_X} \mu^X(m_X) = w$, where $\mu^a := \texttt{DA}(\succ^a)$. Since $\mu^a(m_X) \neq w$ by the contradiction assumption, we further obtain that $\mu^a(m_X) \succ_{m_X} w$.

Now, consider the no-regret profiles $\succ^b, \succ^c, \succ^d, \ldots$ (as established in Lemma 9). Suppose $\mu^X(m_X) \succ_{m_X} \mu^b(m_X)$, where $\mu^b := \texttt{DA}(\succ^b)$. Then, since men propose in decreasing order of their preference, $(m_X, \mu^b(m_X)) \in P_{\succ^b \setminus \succ^X}$. Lemma 12 consequently implies that $(m_X, \mu^b(m_X)) \in P_{\succ}$, and thus $\mu^b(m_X) \succeq_{m_X} \mu(m_X)$. Since $\mu(m_X) \succeq_{m_X} \mu^b(m_X)$ by Corollary 1, we infer that $\mu^b(m_X) = \mu(m_X)$. This, combined with $\mu^X(m_X) \succ_{m_X} \mu^b(m_X)$, gets us $w = \mu^X(m_X) \succ_{m_X} \mu(m_X)$. From the accomplice manipulation assumption, we also know that $m_X = \mu^X(w) \succ_w \mu(w)$. However, this implies that the pair (m_X, w) blocks μ with respect to $\succ$, posing a contradiction. Thus, we have shown that $w = \mu^X(m_X) \not\succ_{m_X} \mu^b(m_X)$. Since $\mu^b(m_X) \neq w$ by the initial contradiction assumption, we further obtain that $\mu^b(m_X) \succ_{m_X} w$. By a similar argument, we get that the women $\mu^c(m_X), \mu^d(m_X), \ldots$ are also placed above w in the list $\succ_{m_X}$.

Since m_X is matched with w under μ^X, he must propose to w during the execution of DA algorithm on $\succ^X$, i.e., $(m_X, w) \in P_{\succ^X}$. From Lemma 14, we have that $(m_X, w) \in P_{\succ^a} \cup P_{\succ^b} \cup P_{\succ^c} \ldots$. Since m_X is a truthful agent, his preference list remains unchanged, and therefore he ranks the women $\mu^a(m), \mu^b(m_X), \mu^c(m_X), \ldots$ strictly above w under each of the profiles $\succ^a, \succ^b, \succ^c, \ldots$. This, however, poses a contradiction since men propose in decreasing order of their preference. □

We are now ready to prove Theorem 5.2.

THEOREM 5.2. *If there is an optimal with-regret accomplice manipulation, then there is an optimal inconspicuous with-regret accomplice manipulation.*

PROOF. From Proposition 3 (and subsequent remarks), we know that any accomplice manipulation can be simulated via push up and push down operations. Lemma 2 shows that any beneficial manipulation that is achieved by some combination of pushing up a set $X \subseteq W$ and pushing down $Y \subseteq W$ can be weakly improved by only pushing up $X \subseteq W$. Finally, from Lemma 4, we know that any match for the manipulating woman w that is achieved by pushing up $X \subseteq W$ is also achieved by pushing up exactly one woman in X, thus establishing the desired inconspicuousness property. □

10.2 Proof of Corollary 4

Corollary 4. *An optimal with-regret accomplice manipulation strategy can be computed in $O(n^3)$ time.*

PROOF. (sketch) The algorithm simply promotes each woman that is below $\mu(m)$ in the accomplice's true preference list to some position above $\mu(m)$ and checks the DA outcome. The total number of such checks is $O(n)$, and for each check, running the DA algorithm takes $O(n^2)$ time. □

11 OMITTED MATERIAL FROM SECTION 6

Recall from our previous experiments that we generated 1000 profiles uniformly at random for each value of $n \in 3, ..., 40$ (where n is the number of men/women) and allowed any man to be chosen as an accomplice. We follow the same setup for all subsequent experiments unless stated otherwise.

11.1 Fraction of Women Who Improve (Cont'd)

We revisit the experiment in which we compared the fraction of women who are able to improve through no-regret accomplice and self manipulation. In Table 1, we catalog the number of women who benefit from both strategies when $n = 20$. Not only are there more instances where at least one woman improves through no-regret accomplice manipulation, but there are also more instances where larger numbers of women improve. For example, there are no instances where more than ten women improve through self manipulation. This is a stark contrast to no-regret accomplice manipulation through which sixteen women are able to improve in one of the instances. Interestingly, there are no instances where *exactly one* woman improves through no-regret accomplice manipulation. This is due to the sequence of proposals that occur after a no-regret push up operation. We formalize this observation in Proposition 6.

PROPOSITION 6. *Let $\succ$ be a preference profile and $\mu := \texttt{DA}(\succ)$. For any man m, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$ and $\mu' := \texttt{DA}(\succ')$. If m does not incur regret and $\mu' \neq \mu$, then there exist at least two distinct women $w', w'' \in W$ such that $\mu'(w') \succ_{w'} \mu(w')$ and $\mu'(w'') \succ_{w''} \mu(w'')$, and at least two distinct men $m', m'' \in M$ such that $\mu(m') \succ_{m'} \mu'(m')$ and $\mu(m'') \succ_{m''} \mu'(m'')$.*

Before proving Proposition 6, we show that if a man m performs a no-regret push up operation such that all the women he pushes up prefer him less than their original DA partners, then the push up operation is *weak* (i.e., the manipulated matching is the same as the original matching).

Lemma 15. *Let $\succ$ be a preference profile and $\mu := \texttt{DA}(\succ)$. For any man m, let $\succ' = \{\succ_{-m}, \succ_m^{X\uparrow}\}$ and $\mu' = \texttt{DA}(\succ')$ such that m does not incur regret. If $\mu(w') \succ_{w'} m$ for all $w' \in X$, then $\mu' = \mu$.*

PROOF. Since m does not incur regret after a push up operation, μ' must be stable with respect to the true preferences, i.e., $\mu' \in S_{\succ}$ (Theorem 4.2). Additionally, Corollary 1 implies that μ' is weakly better for all women (i.e., $\mu' \succeq_W \mu$) and weakly worse for all men (i.e., $\mu \succeq_M \mu'$).

Suppose, for contradiction, that $\mu' \neq \mu$. Let $Z := \{z \in W : \mu'(z) \succ_z \mu(z)\}$ denote the set of women with a strictly more preferable partner under μ'. Thus, by assumption, $Z \neq \emptyset$. Let $Y := \{y \in M : \mu'(y) \in Z\}$ denote the set of men whose μ'-partners are in the set Z. Note that any man not in Y has the same partner under μ and μ'; in particular, $m \notin Y$ by the no-regret assumption. Also note that each man in Y strictly prefers its partner under μ than under μ', i.e., $\mu \succ_Y \mu'$ (this is an easy consequence of the stability of μ' with respect to the true profile $\succ$).

Consider the execution of the DA algorithm on the profile $\succ'$. Since the men propose in decreasing order of their preference, each man in Y must be rejected by his μ-partner during the algorithm. Let $m_1 \in Y$ denote the man who is the *earliest* to be rejected by his μ-partner (i.e., the woman he is matched to under the matching μ), say $w_1 := \mu(m_1)$. Then, w_1 must have at hand a more preferable proposal, say m_2 (i.e., $m_2 \succ_{w_1} m_1$), that she does not receive under the execution of $\texttt{DA}(\succ)$.

No. of women who benefit	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No. of instances (accomplice)	307	0	101	102	88	79	88	59	49	54	28	21	11	8	3	1	1	0	0	0	0
No. of instances (self)	411	151	178	128	68	33	20	7	1	2	1	0	0	0	0	0	0	0	0	0	0

Table 1: The number of instances (out of 1000) where a varying numbers of women benefit through no-regret accomplice manipulation and self manipulation when $n = 20$.

Since m_2 is not rejected by his μ-partner yet, we have that $w_1 \succ'_{m_2} \mu(m_2)$. Now, if $m_2 \neq m$, then we get $w_1 \succ_{m_2} \mu(m_2)$, which contradicts the stability of μ with respect to $\succ$ as (m_2, w_2) would constitute a blocking pair. Otherwise, if $m_2 = m$, then it must be that $w_1 \in X$ (since the women in X are the only ones who m proposes to under $\succ'$ but not under $\succ$). Then, from the above condition, we will have $m = m_2 \succ_{w'} m_1 = \mu(w')$ for some $w' \in X$, which contradicts the condition given in the lemma. Hence, $\mu = \mu'$, as desired. □

We are now ready to prove Proposition 6.

Proof of Proposition 6. Let $X \subseteq W$ be the set of women that m pushes up. Since it is assumed that $\mu' \neq \mu$, the contrapositive of Lemma 15 implies there exists a woman $w' \in X$ such that $\mu(w') \not\succ_{w'} m$. By the push up assumption, $\mu(m) \neq w'$; otherwise m would not have been able to push up w'. This implies that $m \neq \mu(w')$ and thus, $m \succ_{w'} \mu(w')$. Since men propose in decreasing order, m proposes to w' under $\texttt{DA}(\succ')$. By no-regret assumption $\mu(m) = \mu'(m)$, we know that w' rejects m at some point to be matched with another man m' such that $m' \succ_{w'} \mu(w')$. This implies that m' did not propose to w' under $\texttt{DA}(\succ)$, and thus, m' is matched to a more preferred woman under $\succ$, i.e., $\mu(m') \succ_{m'} \mu'(m')$.

Now let $\hat{m} := \mu(w')$ be w''s partner under $\succ$. By Corollary 1, it must be the case that $\mu(\hat{m}) \succ_{\hat{m}} \mu'(\hat{m})$. Let $w'' := \mu'(\hat{m})$. Following the same reasoning as above, since $\mu(m) = \mu'(m)$ and $m \neq \mu(w'')$, we have $m \succ_{w''} \mu(w'')$. The order of proposals under $\succ'$ indicates that w'' rejects m's proposal to be matched to a more preferred man m'' under $\texttt{DA}(\succ')$, which consequently implies that $\mu'(w'') \succ_{w''} \mu(w'')$ while $\mu(m'') \succ_{m''} \mu(m'')$. Therefore, w' and w'''s partners are strictly improved whereas m' and m'''s partners are strictly worsened off under $\texttt{DA}(\succ')$. □

In the same experiment, we additionally computed the fraction of instances in which it is possible for at least one woman to improve through no-regret accomplice and self manipulation. The results in Figure 6 once again suggest that no-regret accomplice strategies are more prevalent than self manipulation.

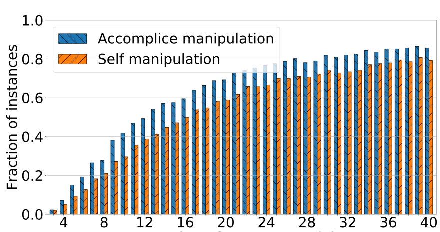

Figure 6: Comparing fractions of instances that are manipulable by at least one woman through no-regret accomplice and self manipulation.

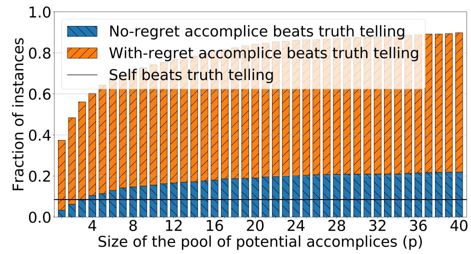

Figure 7: Comparing no-regret accomplice, with-regret accomplice (with variable-sized pools of potential accomplices for both), and self manipulation against truthful reporting when $n = 40$.

11.2 How Much Flexibility is Really Needed in Choosing the Accomplice?

So far, our experiments have taken the optimistic approach of allowing the strategic woman to pick *any* man of her choice as the accomplice. To examine the exact extent of flexibility that this assumption requires, we conduct an experiment where the accomplice is chosen from a *fixed* pool of p men for some $p \leq n$ (for example, when $p = 5$, we pick the best accomplice from a fixed set of five men). For $n = 40$, we ran the no-regret accomplice, with-regret accomplice, and self manipulation strategies on 1000 profiles for $p \in \{1, \ldots, 40\}$. The results are presented in Figure 7.

One would expect with-regret accomplice manipulation to outperform self manipulation when there is variability in accomplices. Indeed, the strategic woman could simply ask her top choice man to place her at the top of his list. However, we observe that, in expectation, with-regret accomplice manipulation outperforms self manipulation for *every* pool size p (thus, a with-regret strategy outperforms self manipulation even when a single accomplice is randomly chosen in advance). Furthermore, the comparatively limited no-regret accomplice manipulation is also, on average, better than self manipulation when there are at least four men to choose from. These observations suggest that the superior performance of accomplice manipulation can be achieved even with a modest amount of flexibility in the choice of the accomplice.

11.3 Regret vs. Improvement

We examine the tradeoff between regret (of the accomplice) and improvement (of the strategic woman) under the with-regret manipulation model. Rather than allowing any man to be chosen as an accomplice, we ran the with-regret manipulation strategy on a fixed woman w and recorded the outcomes after individually using each man as an accomplice. In other words, if there were multiple optimal strategies for w, we chose the one that caused the accomplice to incur the least amount of regret. Figure 8 illustrates the distribution of

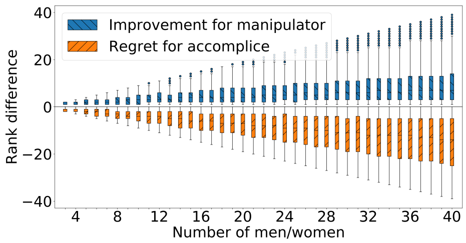

Figure 8: Comparing distributions of improvement for the strategic woman w and regret for the accomplices. The solid bars, whiskers, and dots denote the interquartile range, range excluding outliers, and outliers, respectively.

improvement achieved for w and regret incurred by the accomplices in terms of the difference in the ranks of their matched partners before and after manipulation. Interestingly, the expected regret for the accomplice is greater than the expected improvement for the manipulating woman.

12 SUBOPTIMAL ACCOMPLICE MANIPULATION

We have already shown that any *optimal* accomplice manipulation strategy admits an equivalent inconspicuous strategy (Theorems 4.3 and 5.2). Theorem 12.1 strengthens this result by showing that *any* beneficial (i.e., optimal or suboptimal) accomplice manipulation admits an equivalent inconspicuous strategy for both the no-regret and with-regret settings. In order to prove this, we start by introducing some new results (Lemmas 16 and 17).

Lemma 16. *Let (m, w) be a manipulating pair. Let X be a set of women who m can push up and $\mu(w)$ be w's match after m pushes up all women in X. Let $w' \in X$ be the single woman m needs to push up to get w matched with $\mu(w)$ (as guaranteed by Lemmas 3 and 4). Then, m can push up any subset of women in X that contains w' (i.e., any subset $S \subseteq X$ such that $w' \in S$) to get w matched with $\mu(w)$.*

Proof. Let $\succ^X := \{\succ_{-m}, \succ_m^{X\uparrow}\}$, $\succ^S := \{\succ_{-m}, \succ_m^{S\uparrow}\}$, and $\succ^{w'} := \{\succ_{-m}, \succ_m^{w'\uparrow}\}$. Without loss of generality, $\succ^S$ is derived from $\succ^X$ if m pushes down the set $X \setminus S$. Similarly, $\succ^{w'}$ is derived from $\succ^S$ if m pushes down the set $S \setminus \{w'\}$. From Lemma 1, we get that $\mu^X \succeq_W \mu^S$ and $\mu^S \succeq_W \mu^{w'}$, where $\mu^X := \texttt{DA}(\succ^X)$, $\mu^S := \texttt{DA}(\succ^S)$, and $\mu^{w'} := \texttt{DA}(\succ^{w'})$. Since it is assumed that $\mu^X(w) = \mu^{w'}(w)$, it must be the case that $\mu^S(w) = \mu^X(w)$. Thus, m can push up any subset to get w matched with $\mu(w)$. □

The next result (Lemma 17) shows that a strictly beneficial accomplice manipulation that uses a combination of push up and push down operations can be achieved through push up operations alone. This strengthens Lemma 2 which showed that a combination of push up and push down operations is *weakly worse* than push up operations alone.

Lemma 17. *Let (m, w) be a manipulating pair, and let $\succ$ be a preference profile. For any subsets of women $X \subseteq W$ and $Y \subseteq W$, let $\succ' := \{\succ_{-m}, \succ_m^{X\uparrow}\}$ denote the preference profile after pushing up the set X and $\succ'' := \{\succ_{-m}, \succ_m^{X\uparrow, Y\downarrow}\}$ denote the profile after pushing up X and pushing down Y in the true preference list $\succ_m$ of man m. Let $\mu := \texttt{DA}(\succ)$, $\mu' := \texttt{DA}(\succ')$, and $\mu'' := \texttt{DA}(\succ'')$. If $\mu''(w) \succ_w \mu(w)$, then $\mu''(w) = \mu'(w)$.*

Proof. Suppose, for contradiction, that $\mu''(w) \neq \mu'(w)$. This, combined with $\mu'(w) \succeq_w \mu''(w)$ (Lemma 2), gets us $\mu'(w) \succ_w \mu''(w)$. Additionally, we have assumed that $\mu''(w) \succ_w \mu(w)$, which gets us $\mu'(w) \succ_w \mu(w)$.

$\succ_m$:	$\cdots$	Y	$\cdots$	$\mu(m)$	$\cdots$	X	$\cdots$
$\succ'_m$:	$\cdots$	X	$\cdots$	Y	$\cdots$	$\mu(m)$	$\cdots$
$\succ''_m$:	$\cdots$	X	$\cdots$	$\mu(m)$	$\cdots$	Y	$\cdots$
$\succ^*_m$:	$\cdots$	$\mu(m)$	$\cdots$	X	$\cdots$	Y	$\cdots$

Figure 9: An illustration of man m's preference lists under the profiles $\succ$, $\succ^*$, $\succ'$, and $\succ''$ in the proof of Lemma 17.

Now, let $Z := X \cup Y$. Consider a preference profile $\succ^*$ derived from the true profile $\succ$ by pushing down all women in Z below $\mu(m)$ in the accomplice m's true preference list (see Figure 9). From Proposition 4, we have that $\mu^*(m) = \mu(m)$, where $\mu^* := \texttt{DA}(\succ^*)$. This implies that a push up/down operation with respect to $\mu^*(m)$ is equivalent to the same operation with respect to $\mu(m)$. Therefore, starting with $\succ^*_m$, if m pushes up all women in Z, then we obtain the profile $\succ'$. Similarly, starting with $\succ^*_m$, if m instead pushes up all women in X (respectively, Y), then we obtain the profile $\succ''$ (respectively, $\succ$).

Since profile $\succ'$ is derived from $\succ^*$ via a push up operation of the set Z, Lemma 3 implies that the same match for w, namely $\mu'(w)$, can be achieved by promoting just one woman, say $w' \in Z$, in the preference list $\succ'_m$. Since X and Y are disjoint sets, we have that either $w' \in X$ or $w' \in Y$. If $w' \in X$, then from Lemma 16, we have that $\mu''(w) = \mu'(w)$, contradicting our original assumption. On the other hand, if $w' \in Y$, then again from Lemma 16 we get $\mu(w) = \mu'(w)$, contradicting the condition $\mu'(w) \succ_w \mu(w)$ that we showed above. □

Theorem 12.1. *Any beneficial accomplice manipulation is, without loss of generality, inconspicuous.*

Proof. From Proposition 3 (and the subsequent remarks), we know that any accomplice manipulation can be simulated via push up and push down operations. Lemma 17 shows that *any* beneficial (i.e., optimal or suboptimal) manipulation that is achieved by some combination of pushing up a set $X \subseteq W$ and pushing down $Y \subseteq W$ can also be achieved by only pushing up $X \subseteq W$. Finally, from Lemma 3, we know that any match for the manipulating woman w that is achieved by pushing up $X \subseteq W$ is also achieved by pushing up exactly one woman in X, thus establishing the desired inconspicuousness property. □